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APPLICATION OF THE PHASE INTEGRAL METHOD TO THE ANALYSIS OF THE DIFFRACTION AND REFRACTION OF WIRELESS WAVES ROUND THE EARTH

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1. Introduction

The increasing use of short waves of less than 10 m. has given a new stimulus to the problem of calculating the signal strength to be expected at a distance from a transmitter, and especially of determining the gain of signal strength with height above the ground. For short waves the predominant factor quite near to the transmitter is the diffraction of energy round the curve of the earth with heavy earth losses, so that the problem has to be approached by considering the complete solution for propagation over an imperfectly conducting curved earth.

The solution of the problem was first put on an unimpeachable basis by Watson (1918), who expressed the expansion for the potential function as a contour integral leading to a more rapidly convergent series, and his work has been the starting-point for subsequent writers. His solution is formally complete, but the application of it to practical cases involves considerable mathematical difficulties. He gave the solution explicitly only for the case of long waves where the earth was so highly conducting that, for mathematical purposes, it approached the limit given by a perfect conductor. Also, without further reduction, the solution only expressed the potential function, and hence the electromagnetic field, at points on the surface of the earth.

The complete solution of the problem, especially in its application to the propagation of ultra-short electromagnetic waves, requires the working out of the height factor, and the redetermination of the solution in the case where the simplifying conditions involved in the assumption of long waves and a highly conducting earth are waived.

This solution might be effected by the same methods as were originally used by Watson, in which he transforms the series for the potential function ψ, too slowly convergent to be of any use, into a contour integral which, finally, is equivalent to the sum of the residues of this integral. The calculation of these residues requires the determination of the zeroes of the denominator in the expression

$$\frac{2\pi}{kab}\,\frac{\nu P_{\nu-\frac{1}{2}}(-\mu)\,.\,\zeta_{\nu-\frac{1}{2}}(kb)}{\cos\nu\pi[\partial\zeta'_{s-\frac{1}{2}}(ka)\,.\,\phi(s)/\partial s]_{s=\nu}},\tag{1·1}$$

in which $\phi(s)$ considered as a function of s is given by

$$\phi(s)=1-\frac{\beta k_\iota}{\beta_\iota k}\{\psi'_{s-\frac{1}{2}}(k_\iota a)/\psi_{s-\frac{1}{2}}(k_\iota a)\}\{\zeta_{s-\frac{1}{2}}(ka)/\zeta'_{s-\frac{1}{2}}(ka)\}.\tag{1·2}$$

Watson's original paper should be consulted for the full explanation of these terms. The difficulty of the analysis lies in the fact that this determination of the zeroes requires an intimate knowledge of the intricate Bessel and Hankel functions involved.

In a paper by one of the writers (Eckersley 1931–2) it was shown that Watson's solution was essentially equivalent to an eigenfunction solution, appropriate to wave problems of this type. Thus the eigenfunctions are the residues of the integrals discussed above, and the proper values are the discrete values of s which make the expression for $\phi(s)$ zero. This general relation was made clear by assuming a solution for the potential function ψ of the form $\exp(2\pi\iota S)/r$, substituting this in the general differential equation and determining the approximate condition for S in the form

$$S = \pm\frac{1}{\lambda}\int\left\{1-\left(\frac{\lambda}{2\pi r}\right)^2 n(n+1)\right\}^{\frac{1}{2}} dr, \tag{1·3}$$

where λ is the wave-length, and n the prope rvalues, corresponding to Watson's s values, which are to be determined. On integrating round the branch points of the integrand and equating to an integer, we can determine the discrete values of n, which, except for a certain constant, are identical in form and value with the proper values determined by Watson's full analysis. Watson's solution is then actually, although not explicitly stated so by him, a direct determination of the proper values of eigenfunctions appropriate to the particular problem. The complete equivalence will become quite clear as the analysis proceeds.

The phase integral method, which will be readily recognized as the approximate phase integral method of the Bohr-Sommerfeld theory, gives an alternative method of attacking the problem, and is one which agrees with Watson's theory in the case already dealt with by him. It can, moreover, readily be extended to the case of finite earth conductivity and elevated transmitter and receiver with which we wish to deal. The present paper is concerned with this analysis.

This phase integral method is more than a mathematical artifice to obtain a formula for calculating the electric field at a distance from a wireless transmitter. It has a certain generality, and exhibits the solution as one of a general class (eigen value solutions) appropriate to problems of this kind, encouraging the hope that other problems of this type may be solved by similar methods. As will be seen, it expresses in its form the physical realities of the problem. It affords also a solution to the case not considered in the original problem, but of considerable practical importance, where there is a gradient of refractive index in the atmosphere above the earth.

If r is the distance of the receiver measured from the centre of the earth, and ϕ is its angular distance from the transmitter, the phase integral method expresses the potential function ψ at the receiver in the form

$$\psi = \sum_{n_s} A_s f(r, n_s, \phi) \quad s = 0, 1, 2, \text{etc.}, \tag{1·4}$$

which, as far as the function form $f(r, n_s, \phi)$ is concerned, is identical with appropriate approximations to Watson's eigenfunctions. Only the values of the constants A_s (independent of r and ϕ) are undetermined, but they can be determined by comparison with Watson's analysis, when the proper values n_s appropriate to the case considered have been found. Thus by combining this theory with Watson's original analysis, a complete solution of the problem can be obtained.

During the course of these investigations two papers by Wwedensky (1935–6) appeared, in which the original Watson analysis was extended to take account of the finite conductivity of the earth, and of the effect of elevating the transmitter and receiver. On comparison with our results, it is evident that, with his approximations, his results are strictly equivalent to ours. Slight differences in the numerical results may be attributed to differences in the approximations for determining the proper values. In our method an adjustment was made to bring the proper values to the same as those in Watson's theory, in the limiting case where the conductivity or wave-length is great enough. In his case the limiting values of n differ slightly from Watson's values.

2. General outline of the analysis

It has already been shown (Eckersley 1931–2) that if in the usual way the solution of the fundamental wave equation for the potential function ψ, expressed in spherical co-ordinates, is assumed to be of the form

$$\psi = \psi_1(r)\,\psi_2(\phi), \tag{2·1}$$

where $\psi_1(r)$ is a function of r only, and $\psi_2(\phi)$ is a function of ϕ only, then the value of $\psi_2(\phi)$ is

$$\psi_2(\phi) = P_n(-\cos\phi), \tag{2·2}$$

which is a Legendre polynomial, where n is the proper value to be determined.

Also by writing
$$\psi_1(r) = \frac{1}{r}\exp[2\pi\iota S],$$

then approximately
$$\frac{\partial S}{\partial r} = \pm\frac{V}{\lambda}, \tag{2·3}$$

and
$$S = \pm\frac{1}{\lambda}\int V dr, \tag{2·4}$$

where
$$V = \left[1-g(r)-\left(\frac{\lambda}{2\pi r}\right)^2 n(n+1)\right]^{\frac{1}{2}}. \tag{2·5}$$

The extra term $-g(r)$ is introduced into the expression for S in (1·3), to allow for the refractive index μ of the medium in which the waves are travelling. It was shown that

$$\mu^2 = 1-g(r), \tag{2·6}$$

and that the solution expressed in (2·3) is justified everywhere, except near to the zero of V given in (2·5), i.e. near the branch point of the integrand in (2·4). This exception means that there is an essential error in the phase integral, which in this case gives zero for the integration round the branch point, whereas by comparison with Watson's theory it should be $\frac{1}{2}\pi$. This is introduced into the analysis below as the unknown phase Ω, which can be determined as above, or at a later stage, where the height factor is compared with Debye's approximate formula for the Hankel function.

As a second approximation for $\psi_1(r)$, Jeffreys (1925) has given the value

$$\psi_1(r) = \frac{1}{rf(r)} \exp[2\pi\iota S], \tag{2·7}$$

where S is still determined as before, and where

$$f(r) = V^{\frac{1}{2}}. \tag{2·8}$$

The proper values for n are obtained by the phase integral

$$2\pi \oint \frac{\partial S}{\partial r} dr = 2\pi s + \Omega + \chi, \tag{2·9}$$

where s is a positive integer, and the integral is taken round a contour between $r = r_0$, at the surface of the earth, and $r = r_1$, where r_1 is the branch point at which V is zero.

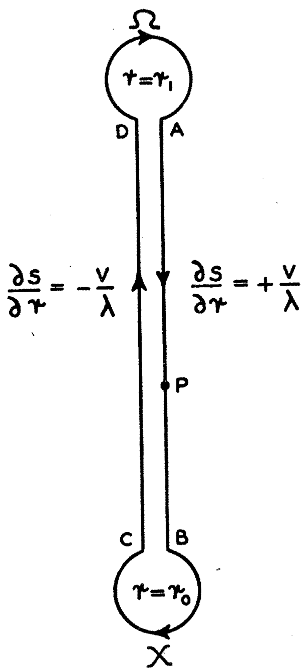


FIG. 1

In fig. 1 AB represents the downcoming branch, which on reflexion at the surface of the earth with a change of phase χ, becomes the upgoing branch CD. The change of phase Ω on integration round the branch point is defined as the change of phase in going from D to A. Inspection shows that for an assumed time factor $\exp(\iota\omega t)$, as is implicit in Watson's analysis, the downcoming branch AB corresponds to $\partial S/\partial r = +V/\lambda$ and the upgoing branch CD to $\partial S/\partial r = -V/\lambda$, while if we start from an arbitrary point P on the branch AB, then positive values of s correspond to integration round the contour in the direction $PADCBP$. If with Wwedensky we were to take the time factor $\exp(-\iota\omega t)$, the upgoing and downcoming branches would become $\partial S/\partial r = +V/\lambda$ and $-V/\lambda$ respectively, and for positive values of s the direction of integration would be $PBCDAP$. It is because we have chosen the Watson form that Ω and χ appear with positive signs on the right-hand side of (2·9).

The phase integral in (2·9) may now be written

$$2\frac{2\pi}{\lambda}\int_{r_0}^{r_1} V dr = 2\pi s + \Omega + \chi, \tag{2·10}$$

where $s = 0, 1, 2, 3$, etc.

The condition $V = 0$ when $r = r_1$ gives from (2·5)

$$1 - g(r_1) - \left(\frac{\lambda}{2\pi r_1}\right)^2 n(n+1) = 0. \tag{2·11}$$

The values of n obtained by solving these two equations correspond to the proper values determined by Watson's analysis, and are put into (2·2) and (2·7) to give the various terms of ψ in (2·1). The $\psi_2(\phi)$ term gives the form of the exponential attenuation factor, and the $\psi_1(r)$ term gives the required height factor.

The phase integral theory also allows for the investigation of the effect of the refractive index of the air, by assigning a value to the function $g(r)$ in (2·6). In general the integrand V becomes very complex, but it is possible to choose the form of $g(r)$ to make the integral tractable, while corresponding to the practical conditions of a refractive index decreasing approximately linearly with height.

The analysis is first worked out for $g(r)$ assumed zero, and the simple diffraction case, in which the effect of the ionosphere as well as of air refraction is ignored, is extended in detail to the study of height-gain effects on short waves.

3. The proper value relations

The notation previously used (Eckersley 1931–2) in deriving the proper value relations is not convenient for our further applications. A slightly modified derivation will therefore be briefly given here.

On putting $g(r) = 0$, V in (2·5) becomes

$$V = \left[1 - \left(\frac{\lambda}{2\pi r}\right)^2 n(n+1)\right]^{\frac{1}{2}}, \tag{3·1}$$

and from (2·11)

$$r_1 = \frac{\lambda}{2\pi}\sqrt{\{n(n+1)\}}. \tag{3·2}$$

Thus (3·1) may be written

$$V = \left[1 - \left(\frac{r_1}{r}\right)^2\right]^{\frac{1}{2}}. \tag{3·3}$$

It is not immediately obvious, but in the sequel it appears that r_1 is only slightly greater than r_0. Anticipating this result we may write

$$r_1 = r_0(1+\xi), \tag{3·4}$$

where ξ is assumed tentatively to be a small quantity, which is in general complex.

(3·2) may now be written

$$\sqrt{\{n(n+1)\}} = 2\pi r_0(1+\xi)/\lambda = x(1+\xi),$$

where

$$x = 2\pi r_0/\lambda. \tag{3·5}$$

In practice, even for long waves, x is very large, so that n can be assumed large, and we may write

$$n+\tfrac{1}{2} = x(1+\xi) = 2\pi r_1/\lambda. \tag{3·6}$$

If now we write
$$\frac{r_1}{r} = \cos\alpha = \frac{r_0}{r}(1+\xi), \tag{3·7}$$

then from (3·3)
$$V = \sin\alpha,$$

and $dr = r_1 \sec\alpha \tan\alpha\, d\alpha$, so that the phase integral in (2·10) becomes

$$2\frac{2\pi}{\lambda}\int_{\alpha_0}^{0} r_1 \tan^2\alpha\, d\alpha = 2\pi s + \Omega + \chi, \tag{3·8}$$

where α_0 is the value of α when $r = r_0$, i.e. from (3·7)

$$\cos\alpha_0 = 1+\xi. \tag{3·9}$$

Since ξ is small, α_0 is small, so that (3·9) gives

$$\xi = -\tfrac{1}{2}\alpha_0^2. \tag{3·10}$$

From (3·8) we have
$$2\left(\frac{2\pi r_1}{\lambda}\right)[-\tan\alpha_0+\alpha_0] = 2\pi s+\Omega+\chi,$$

and since α_0 is small, and $2\pi r_1/\lambda \risingdotseq 2\pi r_0/\lambda = x$ from (3·5),

$$-\frac{2x\alpha_0^3}{3} = 2\pi s+\Omega+\chi. \tag{3·11}$$

This equation has to be solved for α_0, for substitution in (3·10) to give ξ. Of the three roots only one corresponds to our physical problem. Returning to the form of $\psi_2(\phi)$ in (2·2), and replacing the Legendre polynomial, since n is large, by Laplace's approximation, we have

$$\psi_2(\phi) = \left(\frac{2}{\pi n \sin\phi}\right)^{\frac{1}{2}} \cos[(n+\tfrac{1}{2})(\pi-\phi)-\tfrac{1}{4}\pi],$$

provided ϕ is not near to 0 or π.

As far as the modification to Watson's formula is concerned, we need only consider the relevant exponential term of this expression, and remembering that the time factor assumed is $\exp(\iota\omega t)$, we may write

$$\psi_2(\phi) = \exp[-\iota(n+\tfrac{1}{2})\phi]. \tag{3·12}$$

Now from (3·6) and (3·10)

$$-\iota(n+\tfrac{1}{2})\phi = -\iota x\phi[1+\xi] = -\iota x\phi + \frac{\iota x\alpha_0^2}{2}\phi,$$

and the complete wave function is of the form

$$\exp[\iota(\omega t - x\phi)]\exp\left[\frac{\iota x\alpha_0^2}{2}\phi\right]. \tag{3·13}$$

The first term correctly represents a diverging wave, and the second term must contain an attenuation factor. If we write (3·11) as

$$\frac{2x\alpha_0^3}{3} = [2\pi s + \Omega + \chi]\, e^{\iota(2l+1)\pi} \quad l = 0,\ 1,\ \text{or}\ 2,$$

inspection shows that the only root corresponding to an attenuation factor is given by $l = 0$.

Thus $$\alpha_0 = x^{-\frac{1}{3}}[\tfrac{3}{2}(2\pi s + \Omega + \chi)]^{\frac{1}{3}}\, e^{\frac{1}{3}\iota\pi},$$

and $$\xi = -\tfrac{1}{2}\alpha_0^2 = \rho x^{-\frac{2}{3}}\, e^{-\frac{1}{3}\iota\pi}, \tag{3·14}$$

where $$\rho = \tfrac{1}{2}[\tfrac{3}{2}(2\pi s + \Omega + \chi)]^{\frac{2}{3}}. \tag{3·15}$$

ξ is thus indeed small by virtue of the term $x^{-\frac{2}{3}}$, and our tentative assumption is justified.

In terms of ρ we may write α_0 as

$$\alpha_0 = \sqrt{(2\rho)}\, x^{-\frac{1}{3}}\, e^{\frac{1}{3}\iota\pi}. \tag{3·16}$$

We now have from (3·6)

$$n + \tfrac{1}{2} = x(1+\xi) = x + \rho x^{\frac{1}{3}}\, e^{-\frac{1}{3}\iota\pi},$$

which is identical with the form of the proper values found by Watson, and the attenuation factor contained in the second term of (3·13) is

$$\exp[\{\rho x^{\frac{1}{3}}\, e^{-\frac{1}{3}\iota\pi}\}_I \phi] = \exp\left[(2\pi r_0)^{\frac{1}{3}}\, (\rho e^{-\frac{1}{3}\iota\pi})_I \frac{\phi}{\lambda^{\frac{1}{3}}}\right].$$

Writing* $$\rho = |\rho|\, e^{\iota\omega} \tag{3·17}$$

so that from (3·14) $$\xi = |\xi|\, e^{-\iota(\frac{1}{3}\pi - \omega)}, \tag{3·18}$$

the attenuation factor may be written

$$\exp\left[-\frac{\beta\phi}{\lambda^{\frac{1}{3}}}\right],$$

where $$\beta = (2\pi r_0)^{\frac{1}{3}}\, |\rho| \sin(\tfrac{1}{3}\pi - \omega). \tag{3·19}$$

The phase integral theory thus leads directly and simply to the same form of proper value relation as that obtained by Watson, but the relation is also more general, since it takes account of the effect of the finite earth conductivity. The equation for ρ in (3·15) now contains χ, the phase change on reflexion at the surface of the earth, which depends on the value of the earth constants. We pass on now, therefore, to a study of the nature of this dependence.

* ω here should not be confused with ω in the time factor $\exp(\iota\omega t)$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

4. The form of the quantity ρ

Taking for comparison with Watson's form the case of a perfectly conducting earth, we put $\chi = 0$. (3·15) then gives

$$\rho = \frac{1}{2}\left[3\pi\left(s+\frac{\Omega}{2\pi}\right)\right]^{\frac{2}{3}}. \tag{4·1}$$

Now the values of $\frac{1}{2}(3\pi s)^{\frac{2}{3}}$ for $s = 0$, 1 and 2, are each less than the corresponding values for the first three proper values given by Watson, and calculation shows that if Ω is adjusted to make the values of ρ agree with the Watson values, which we will call ρ_0, then Ω is approximately $\frac{1}{2}\pi$, and approaches this value as s is increased. Thus if we write

$$\Omega = 2q\pi, \tag{4·2}$$

so that (4·1) becomes

$$\rho_0 = \tfrac{1}{2}[3\pi(s+q)]^{\frac{2}{3}}, \tag{4·3}$$

and put into this equation the values of ρ_0 given by Watson, we have the following values for q:

s	ρ_0	q
0	0·8083	0·218
1	2·577	0·231
2	3·83	0·248

and as s increases, q approaches $\frac{1}{4}$, and Ω approaches $\frac{1}{2}\pi$.

The phase integral thus suggests a convenient way by which we can adjust the approximate value of ρ_0, obtainable by comparison with the Debye approximations for the Hankel functions, to the true Watson value. Although the validity of so doing may be questionable on strictly analytical grounds, it can be shown that it in no way affects the general form of the analysis which follows. It only gives rise to slight numerical differences from the results obtained by assuming the Debye approximation, and as it has the merit of agreeing with Watson's value in the limit, the adjustment has accordingly been made in computations based on the results of the analysis.

Having determined Ω in this way, the case of the imperfectly conducting earth can be treated by considering the value of χ when the earth has a specific inductive capacity ϵ and a conductivity σ in e.m.u. The value of χ depends upon the angle of incidence of the waves. If this angle is $\frac{1}{2}\pi-\theta$, then χ is found by writing the reflexion coefficient as a complex phase. Thus using a well-known relation for the reflexion coefficient, we have*

$$\frac{(\epsilon-2\iota\sigma\lambda c)\sin\theta-\sqrt{(\epsilon-2\iota\sigma\lambda c-\cos^2\theta)}}{(\epsilon-2\iota\sigma\lambda c)\sin\theta+\sqrt{(\epsilon-2\iota\sigma\lambda c-\cos^2\theta)}} = e^{\iota\chi},$$

which transforms to

$$\chi = 2\tan^{-1}\left[\frac{\iota\sqrt{(\epsilon-2\iota\sigma\lambda c-\cos^2\theta)}}{(\epsilon-2\iota\sigma\lambda c)\sin\theta}\right]. \tag{4·4}$$

* The form $\epsilon-2\iota\sigma\lambda c$ is appropriate to the time factor $\exp(\iota\omega t)$. In the original paper the form $\epsilon+2\iota\sigma\lambda c$ corresponding to $\exp(-\iota\omega t)$ was wrongly used, and e^{χ} was used instead of $e^{\iota\chi}$, thus accounting for the discrepancy between Wwedensky's transition curve and ours.

θ is determined by considering $\cos\theta$, which is the direction cosine of the normal to the wave surface represented by $\psi_2(\phi)$. If d is the distance of the receiver from the transmitter measured along the surface of the earth, we write

$$\psi_2(\phi) = \exp\left[-\frac{2\pi\iota}{\lambda} d\cos\theta\right],$$

and equating this to the form of $\psi_2(\phi)$ in (3·12) we get

$$\frac{2\pi}{\lambda} d\cos\theta = (n+\tfrac{1}{2})\,\phi.$$

But since $d = r_0\phi$, this leads to

$$\cos\theta = (n+\tfrac{1}{2})/x = 1+\xi$$

from (3·6). Thus from (3·9) we see that θ is the same as α_0, and is therefore a small angle.

We can now write (4·4) as
$$\chi = 2\tan^{-1}\eta, \tag{4·5}$$
where
$$\eta = \iota/\zeta\sin\alpha_0 \doteq \iota/\zeta\alpha_0 \tag{4·6}$$
and
$$\zeta = \frac{\epsilon - 2\iota\sigma\lambda c}{\sqrt{(\epsilon - 1 - 2\iota\sigma\lambda c)}}. \tag{4·7}$$

Putting the value of χ in (4·5) into (3·15), and converting into terms of ρ_0 by (4·3), we obtain
$$\rho = \rho_0[1+A\tan^{-1}\eta]^{\frac{2}{3}}, \tag{4·8}$$
where
$$A = 1/\pi(s+q).$$

This is the equation that represents the modification which it is necessary to make to Watson's original ρ_0 values, to take account of the finite conductivity of the earth. In considering it we have to remember that η is itself a function of ρ, since it contains α_0, so that some method has to be found for solving the equation for ρ.

When λ is large, η approaches zero, and ρ approaches ρ_0 as a lower limit. When λ is small, η gets large, and χ approaches π as an upper limit. ρ then approaches an upper limit, say ρ_π, given by putting $\tan^{-1}\eta = \frac{1}{2}\pi$ in (4·8).

Thus
$$\rho_\pi = \rho_0\left[1+\frac{1}{2(s+q)}\right]^{\frac{2}{3}}.$$

Between the two real limits ρ_0 and ρ_π there is a transition region, which has to be investigated by solving (4·8) for ρ.

Under the conditions of our problem we can write without any ambiguity

$$\tan^{-1}\eta = \tfrac{1}{2}\pi - \tan^{-1}\kappa,$$

where
$$\kappa = 1/\eta,$$

so that we can also write (4·8) in the alternative form

$$\rho = \rho_\pi [1 - B \tan^{-1} \kappa]^{\frac{2}{3}}, \tag{4·9}$$

where $$B = 2/\pi[2(s+q)+1].$$

We need now to consider the general proposition of the representation of $\tan^{-1}(Re^{\iota\Phi})$ in the form $X + \iota Y$.

We have $$\tan(X+\iota Y) = Re^{\iota\Phi}.$$

By expanding and equating the real and imaginary parts we get a pair of equations, which can be solved for $\tan X$ and $\tanh Y$ in terms of R and Φ. These solutions can be further transformed by the following substitutions:

Let $$\tan L = 2R\cos\Phi/(1-R^2), \tag{4·10}$$

and $$\sin M = 2R\sin\Phi/(1+R^2), \tag{4·11}$$

then the solutions will be found to be

$$\tan X = \tan \tfrac{1}{2}L, \quad \text{i.e.} \quad X = \tfrac{1}{2}L, \tag{4·12}$$

and $$\tanh Y = \tan\tfrac{1}{2}M, \quad \text{i.e.} \quad Y = \tfrac{1}{2}\log_e[\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}M)]. \tag{4·13}$$

For any given values of R and Φ, L and M can be found from (4·10) and (4·11), and hence X and Y can be found from (4·12) and (4·13). In particular when $R = 1$, $X = \frac{1}{4}\pi$, independently of the value of Φ, and $M = \Phi$, so that

$$\tan^{-1} e^{\iota\Phi} = \tfrac{1}{4}\pi + \iota\tfrac{1}{2}\log_e[\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\Phi)].$$

By using the form for ρ in (4·8) when $|\eta| \leqslant 1$, and the form in (4·9) when $|\kappa| \leqslant 1$, we only need to consider values of R between 0 and 1, and under these conditions Φ always lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. Thus $\tan L$ in (4·10) is always positive, and X is given by $\frac{1}{2}L$, as in (4·12), where L is in the first quadrant, while M in (4·11) always lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

If we put the value of α_0 in (3·16) into the expression for η in (4·6) and if we write ζ in (4·7) as

$$\zeta = |\zeta|\, e^{\iota v},$$

then η may be written, by using (3·17),

$$\eta = \frac{x^{\frac{1}{3}}}{|\zeta|\sqrt{(2|\rho|)}}\, e^{\iota(\frac{1}{6}\pi - v - \frac{1}{2}\omega)}. \tag{4·14}$$

In our problem therefore we have:

for $|\eta| \leqslant 1$ $$R = \frac{x^{\frac{1}{3}}}{|\zeta|\sqrt{(2|\rho|)}}, \tag{4·15}$$

$$\Phi = \tfrac{1}{6}\pi - v - \tfrac{1}{2}\omega, \tag{4·16}$$

and for $|\kappa| \leqslant 1$ $$R = \frac{|\zeta| \sqrt{(2|\rho|)}}{x^{\frac{1}{3}}}, \tag{4·17}$$

$$\Phi = -\tfrac{1}{6}\pi + v + \tfrac{1}{2}\omega. \tag{4·18}$$

When $\epsilon \gg 1$ (4·7) may be written

$$\zeta = \sqrt{(\epsilon - 2\iota\sigma\lambda c)},$$

and when $2\sigma\lambda c \gg \epsilon$ $$|\zeta| = \sqrt{(2\sigma\lambda c)} \quad \text{and} \quad v = -\tfrac{1}{4}\pi.$$

In most practical cases this latter condition holds, and then the relations (4·15) to (4·18) become:

for $|\eta| \leqslant 1$ $$R = \frac{(2\pi r_0)^{\frac{1}{3}}}{\sqrt{(2c)}\sqrt{(2|\rho|)}} \sigma^{-\frac{1}{2}} \lambda^{-\frac{5}{6}}, \tag{4·19}$$

$$\Phi = \tfrac{5}{12}\pi - \tfrac{1}{2}\omega,$$

and for $|\kappa| \leqslant 1$ $$R = \frac{\sqrt{(2c)}\sqrt{(2|\rho|)}}{(2\pi r_0)^{\frac{1}{3}}} \sigma^{\frac{1}{2}} \lambda^{\frac{5}{6}}, \tag{4·20}$$

$$\Phi = -\tfrac{5}{12}\pi + \tfrac{1}{2}\omega.$$

Considering first the η form of ρ, we have from (4·8)

$$\left[\frac{|\rho|}{\rho_0}\right]^{\frac{3}{2}} e^{3\iota\omega/2} = 1 + A(X + \iota Y),$$

so that $$\tan \tfrac{3}{2}\omega = \frac{AY}{1 + AX}$$

and $$\frac{|\rho|}{\rho_0} = [(1 + AX)^2 + (AY)^2]^{\frac{1}{3}}.$$

Similarly for the κ form of ρ from (4·9)

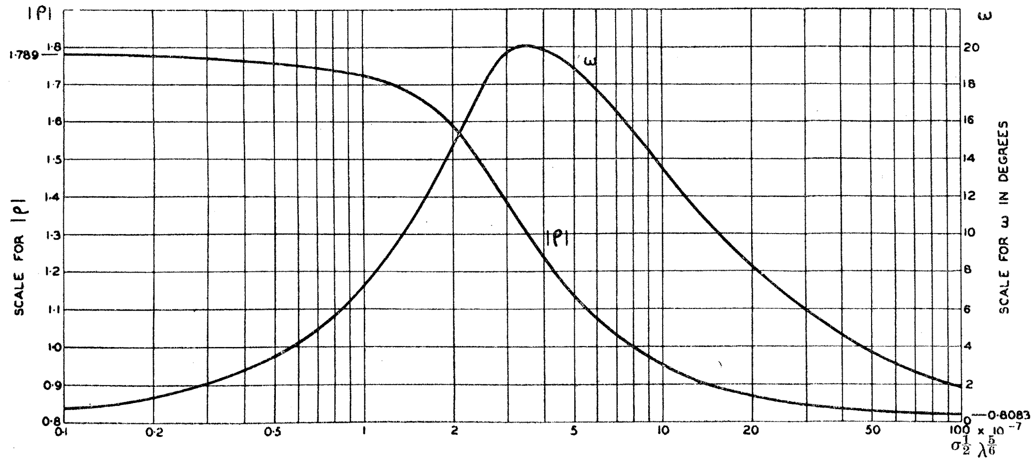
$$\tan \tfrac{3}{2}\omega = \frac{-BY}{1 - BX}$$

and $$\frac{|\rho|}{\rho_\pi} = [(1 - BX)^2 + (BY)^2]^{\frac{1}{3}}.$$

In this case Φ is negative, and hence Y is also negative, so that $\tan \frac{3}{2}\omega$ is positive, and we get continuity at $|\eta| = |\kappa| = 1$ as we should.

To solve these equations we assume a value of R. Now Φ contains ω, and v is also unknown, unless we can assume that it is $-\frac{1}{4}\pi$. As a first approximation therefore we put $\omega = 0$ and $v = -\frac{1}{4}\pi$, so that $\Phi = \pm\frac{5}{12}\pi$, according as we are using the η or the κ form. Having found X and Y by means of the L and M substitutions, we can solve the above equations to get the first approximations to ω and $|\rho|$. By putting the value of $|\rho|$ into R in (4·15) or (4·17) we get a value of $|\zeta|$, and hence of v if we cannot assume that it is $-\frac{1}{4}\pi$. (In this case we must fix values of ϵ and σ, and use the $|\zeta|$ value to find the value of λ corresponding to the chosen value of R.) We can then proceed by successive approximations. In practice, by starting with small values of R, we can take, for any given value of R, the final value of ω found for the previous smaller value of R, as our first approximation in the value of Φ.

When we can assume that $2\sigma\lambda c \gg \epsilon$, we see from (4·19) and (4·20) that for a given value of R there corresponds a value of $\sigma^{\frac{1}{2}}\lambda^{\frac{5}{6}}$, so that $|\rho|$ and ω can be plotted as functions of this quantity. In fig. 2 this has been done, where σ is in e.m.u. and λ is in kms., for $s = 0$ for the first term of the diffraction formula. Actually in practice the condition $2\sigma\lambda c \gg \epsilon$ only breaks down for points for which ρ is already near to the upper limit, and for which ω is very small.



$|\rho|$ and ω for $s = 0$ first term. σ in e.m.u.; λ in kms.

FIG. 2

Having found $|\rho|$ and ω, we can also plot β in (3·19) as a function of $\sigma^{\frac{1}{2}}\lambda^{\frac{5}{6}}$, and this has been done in figs. 3 and 4 for the first two terms of the diffraction formula. We can define the two limits (at which $\omega = 0$) by β_0 and β_π, where

$$\beta_0 = (2\pi r_0)^{\frac{1}{3}}\rho_0 \sin\tfrac{1}{3}\pi$$

and

$$\beta_\pi = (2\pi r_0)^{\frac{1}{3}}\rho_\pi \sin\tfrac{1}{3}\pi,$$

so that

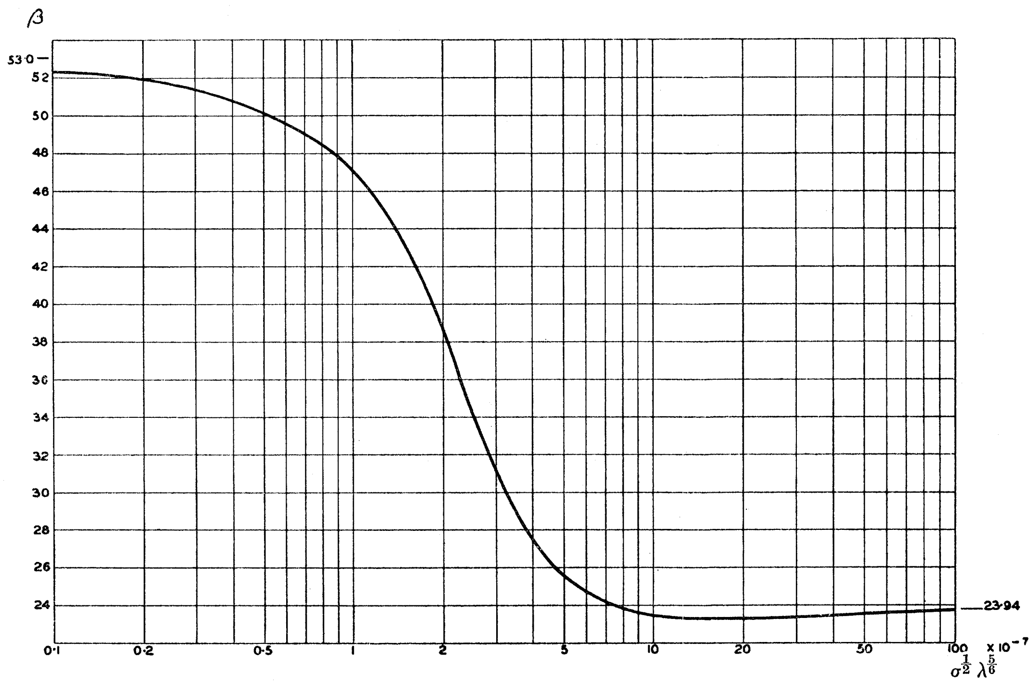
$$\beta = \beta_0 \frac{|\rho|}{\rho_0}\frac{\sin(\frac{1}{3}\pi - \omega)}{\sin\frac{1}{3}\pi} = \beta_\pi \frac{|\rho|}{\rho_\pi}\frac{\sin(\frac{1}{3}\pi - \omega)}{\sin\frac{1}{3}\pi}.$$

It is interesting to note that as $\sigma^{\frac{1}{2}}\lambda^{\frac{5}{6}}$ is decreased, β initially gets smaller than β_0 because, as can be shown analytically, ω always being positive, the term $\frac{\sin(\frac{1}{3}\pi - \omega)}{\sin\frac{1}{3}\pi}$ becomes less than unity, and initially more than counterbalances the increase in the value of $|\rho|/\rho_0$. The effect is only small, but its physical meaning is not obvious. It is possibly an anomalous result, produced by the approximations made in adjusting the limiting value of ρ to the Watson value ρ_0, or in Wwedensky's analysis by the approximations involved in the use of the Debye form of the Hankel functions.*

* The initial drop in the value of β has been found independently by Dr B. Van der Pol and Dr H. Bremmer by a completely different method. Our transition curve agrees closely with the one they have kindly shown us, except for a small difference in the upper limit due to the approximation involved in our assumed value of $q\pi$.

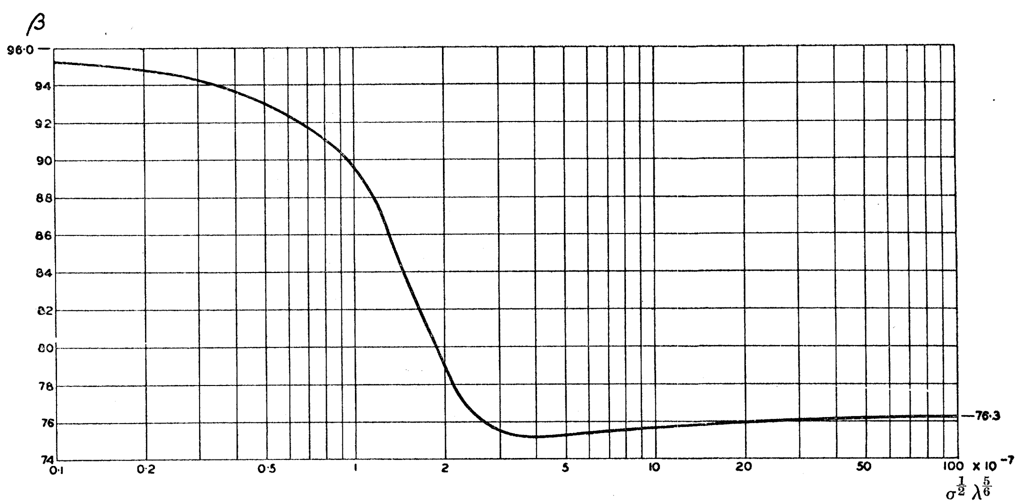
For convenience a summary is here given of the limits of ρ and β for the first three terms:

s	A	B	ρ_0	ρ_π	β_0	β_π	ω_{max}
0	1·460	0·428	0·8083	1·789	23·94	53·0	20°
1	0·258	0·184	2·577	3·24	76·3	96·0	7·5°
2	0·142	0·116	3·83	4·38	113·3	129·6	4·6°



β for $s = 0$ first term. σ in e.m.u.; λ in kms.

FIG. 3



β for $s = 0$ second term. σ in e.m.u.; λ in kms.

FIG. 4

5. The physical significance of the ψ function for points above the surface of the earth

In the above analysis we have determined the form of the diffraction curve for points on the surface of the earth, for which the study of the function $\psi_2(\phi)$ suffices. When we come to consider points above the surface of the earth, we have to take into account the variations of the function $\psi_1(r)$, and to study the value of the whole ψ function given by (2·1).

Returning to the form of $\psi_1(r)$ given in (2·7), and using (2·8) and (2·4), we obtain, on making the same substitutions as in (3·4) and (3·7),

$$\psi_1(r) = \frac{1}{r\sqrt{(\sin\alpha)}}\exp\left[\pm\frac{2\pi\iota}{\lambda}r_1(\tan\alpha-\alpha)\right]. \tag{5·1}$$

The alternative signs in the exponential term correspond to the downcoming and upgoing branches shown in fig. 1. For values of r greater than $|r_1|$ the upgoing branch corresponding to the negative sign is predominant. Thus above $r=|r_1|$ we may write

$$\psi_1(r) = \frac{1}{r\sqrt{(\sin\alpha)}}\exp\left[-\frac{2\pi\iota}{\lambda}r_1(\tan\alpha-\alpha)\right]. \tag{5·2}$$

Referring back to the expression for r_1 in (3·4), and using (3·18), we have

$$r_1 = r_0[1+|\xi|\,e^{-\iota(\frac{1}{3}\pi-\omega)}], \tag{5·3}$$

and since $|\xi|$ is very small r_1 is nearly wholly real, and

$$|r_1| = r_0[1+|\xi|\cos(\tfrac{1}{3}\pi-\omega)]. \tag{5·4}$$

It follows that we may write (5·3) as

$$r_1 = |r_1|\,[1-\iota y], \tag{5·5}$$

where
$$y = |\xi|\sin(\tfrac{1}{3}\pi-\omega)\,\frac{r_0}{|r_1|},$$
which is $\ll 1$.

Now we also note that when r increases beyond $|r_1|$ the angle α rapidly becomes mainly real. For if we write $r=|r_1|+h$, then from (3·7) and (5·5)

$$\cos\alpha = \frac{|r_1|}{|r_1|+h}[1-\iota y].$$

Since α and $h/|r_1|$ are both small, this leads to

$$\alpha = \sqrt{\left(2\left[\frac{h}{|r_1|}+\iota y\right]\right)}. \tag{5·6}$$

As h increases, $h/|r_1|$ becomes great compared with y, and α becomes nearly real, approximating to an angle $|\alpha|$, where

$$\cos|\alpha| = \frac{|r_1|}{r}, \quad \text{and} \quad |\alpha| = \sqrt{\frac{2h}{|r_1|}} \doteqdot \sqrt{\frac{2h}{r_0}}. \tag{5·7}$$

From (5·6) we have $\alpha = |\alpha| + \iota y/|\alpha|$,

so that $r_1\alpha$ is given approximately by

$$r_1\alpha = r_1|\alpha| + \iota y\frac{|r_1|}{|\alpha|}. \tag{5·8}$$

Considering now the value of $r_1 \tan\alpha$, we have from (3·7)

$$r_1\tan\alpha = \sqrt{(r^2 - r_1^2)} \doteqdot \sqrt{\{(r^2 - |r_1|^2) + |r_1|^2 . 2\iota y\}}.$$

But if we write

$$d = |r_1|\tan|\alpha| = \sqrt{(r^2 - |r_1|^2)} \doteqdot \sqrt{(2|r_1|h)} \doteqdot |r_1| . |\alpha|$$

from (5·7), then $$r_1\tan\alpha = d + \iota y\frac{|r_1|^2}{d} = d + \iota y\frac{|r_1|}{|\alpha|}. \tag{5·9}$$

By combining (5·8) and (5·9) we get

$$r_1(\tan\alpha - \alpha) = -r_1|\alpha| + d.$$

Putting this value in $\psi_1(r)$ in (5·2) we obtain

$$\psi_1(r) = \frac{1}{r\sqrt{(\sin\alpha)}}\exp\left[-\frac{2\pi\iota}{\lambda}(-r_1|\alpha| + d)\right].$$

To get the whole potential function this must be multiplied by $\psi_2(\phi)$ as given in (3·12), i.e.

$$\psi_2(\phi) = \exp[-\iota(n+\tfrac{1}{2})\phi] = \exp\left[-\frac{2\pi\iota}{\lambda}r_1\phi\right] \tag{5·10}$$

from (3·6), so that $$\psi = \frac{1}{r\sqrt{(\sin\alpha)}}\exp\left[-\frac{2\pi\iota}{\lambda}r_1(\phi - |\alpha|) - \frac{2\pi\iota}{\lambda}d\right]. \tag{5·11}$$

When put in this form it suggests an immediate geometrical and physical interpretation.

In fig. 5, O represents the centre of the earth, and T is the transmitter on the surface of the earth. R is the receiver at the point $(|r_1|+h, \phi)$, so that $OR = |r_1| + h$ and $\angle TOR = \phi$. PQ represents an arc of radius $|r_1|$, $OP = OQ = |r_1|$, and $QR = h$. If from R a tangent is drawn to the arc PQ to touch it at S, then

$$\cos SOR = \frac{OS}{OR} = \frac{|r_1|}{|r_1| + h},$$

so that $\angle SOR = |\alpha|$, and $RS = |r_1|\tan|\alpha| = d$.

Also $\angle TOS = \phi - |\alpha|$.

Thus in (5·11) the term $\exp\left[-\frac{2\pi\iota}{\lambda}r_1(\phi-|\alpha|)\right]$, by comparison with (5·10), is equivalent to propagation along the arc PS, with the attenuation associated with the bending through the angle TOS, while the term $\exp\left[-\frac{2\pi\iota}{\lambda}d\right]$ is equivalent to unattenuated propagation along the tangent plane from S to R. The condition that $h/|r_1|$ is $\gg y$

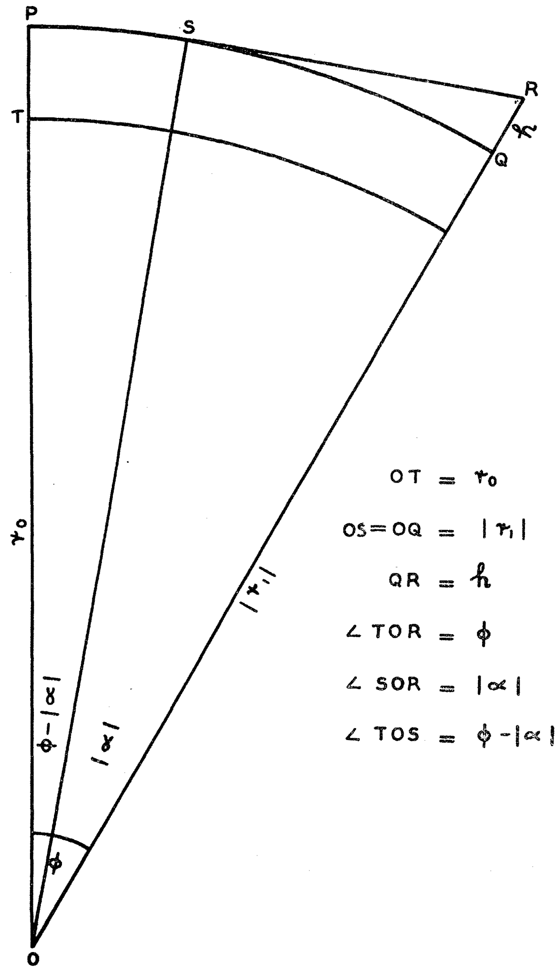


FIG. 5

implies, for instance, that for wave-lengths in the range 2–10 m. h should be greater than, say, 500 m. Thus, provided we are far enough from the transmitter to be able to fulfil this condition, we get a simple physical picture of a process of diffraction followed by an unattenuated path out into space, analogous to the picture one would expect from ordinary optical theory.

But the important point emerges that the process is referred not to the curved surface of the earth, but to a fictitious sphere of radius $|r_1|$ somewhat greater than the radius of the earth. This point is obscured in ordinary optical theory because $|r_1|$ is then so nearly

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equal to r_0, but with ultra-short waves the wave-length has become long enough for $|r_1|$ to have a very important physical significance.

This picture allows us to express an approximate relation between the height of the receiver above $|r_1|$ and the decibel gain in signal strength produced by raising it to this height from $|r_1|$. It will be noted that we are concerned here with the gain above the signal strength level at $|r_1|$, since the whole picture is referred to the sphere of radius $|r_1|$. The variation of signal strength below $|r_1|$, or to put it another way, the gain of signal strength at $|r_1|$ above the level at the surface of the earth, can only be dealt with by the more detailed analysis to follow. The picture shows that the gain in signal strength produced by raising the receiver through a height h from Q to R, is equivalent to the reduction in attenuation produced by moving the receiver along the arc QP to S through an angle $|\alpha|$. Calling the decibel gain D, we have from (3·19)

$$D = (20\log_{10} e)\frac{\beta|\alpha|}{\lambda^{\frac{1}{3}}}$$
$$= (20\log_{10} e)\,|\rho|\sin(\tfrac{1}{3}\pi-\omega)\,.\left(\frac{2\pi r_0}{\lambda}\right)^{\frac{1}{3}}|\alpha|.$$

For ultra-short waves $\omega \doteqdot 0$, and putting in the value of $|\alpha|$ from (5·7) and expressing r_0, λ and h in metres, this reduces to

$$D = 1{\cdot}44\,|\rho|\,h^{\frac{1}{2}}\lambda^{-\frac{1}{3}}. \tag{5·12}$$

If, further, we assume that the first term of the diffraction formula is predominant, and that ρ has the value of its upper limit $\rho_\pi = 1{\cdot}789$, then

$$D = 2{\cdot}58h^{\frac{1}{2}}\lambda^{-\frac{1}{3}}.$$

Thus for h above, say, 500 m., we can regard D as a function of $h\lambda^{-\frac{2}{3}}$, and for a given wave-length D is proportional to $\sqrt{h}$.

The radius $|r_1|$ which plays such an important part in the theory is given by (5·4), and corresponds to a height h_1 above the surface of the earth, where

$$h_1 = |r_1| - r_0 = r_0|\xi|\cos(\tfrac{1}{3}\pi-\omega). \tag{5·13}$$

For short waves we again put $\omega = 0$, and putting in the value of $|\xi|$ derived from (3·14) we obtain

$$h_1 = \tfrac{1}{2}(2\pi)^{-\frac{2}{3}}|\rho|\,r_0^{\frac{1}{3}}\lambda^{\frac{2}{3}}.$$

If r_0 and λ are expressed in metres,

$$h_1 = 27{\cdot}2\,|\rho|\,\lambda^{\frac{2}{3}}\text{ metres}, \tag{5·14}$$

and for ultra-short waves for which $|\rho| = \rho_\pi = 1{\cdot}789$,

$$h_1 = 48{\cdot}7\lambda^{\frac{2}{3}}\text{ metres}. \tag{5·15}$$

The geometrical picture given above may also be interpreted by saying that as we go up from the surface of the earth we do not get out of the diffraction shadow when we reach the edge of the optical shadow represented by the tangent plane to the earth at T, but only when we reach the tangent plane to the sphere of radius $|r_1|$ at a height h_1 above T. Points on this plane correspond to $\phi = |\alpha|$, i.e. to $\phi - |\alpha| = 0$, and therefore to zero attenuation.

As we should expect, h_1 increases with λ. This implies that on long waves, where the attenuation is much less than for short waves, the gain for a given height is less. This is found to follow from the more detailed analysis for points below $|r_1|$. Physically this corresponds to the fact that the edge of the diffraction shadow gets more ill defined as the wave-length increases.

The above argument provides a limit to the practical application of the height-gain analysis. From (5·14) we see that h_1 is proportional to $|\rho|$, and therefore increases for the higher terms of the diffraction formula. For points beyond the tangent plane corresponding to the first term, the higher terms would have to be taken into account. Actually for points well out in space, the vectorial addition of all the terms obtained by the complete analysis should reduce to the simple inverse distance law. Although the analysis which follows has a formal generality, the application of the height-gain theory is confined to the ultra-short waves, for which the results are of great practical importance in the region below the tangent plane, where they can be represented adequately by the first term of the diffraction theory.

6. The detailed height-gain analysis

In the above geometrical picture we limited ourselves to points outside the sphere of radius $|r_1|$, and neglected one of the branches of the wave function given in (5·1). To obtain the picture we also transformed the function $\psi_1(r)$, so that when re-grouped with the value of $\psi_2(\phi)$, it had an immediate physical significance.

In considering the complete form of the height-gain analysis, especially in the important region below $|r_1|$, it is more convenient to keep the $\psi_2(\phi)$ and the $\psi_1(r)$ functions separate, and to regard the $\psi_2(\phi)$ function as giving the form of the attenuation along the surface of the earth, and the $\psi_1(r)$ function as determining the height-gain relation for any given angular distance, which is independent of this distance, provided we are not too near to the transmitter.

The expression for $\psi_1(r)$ in (5·1) may be written

$$\psi_1(r) = \frac{1}{r\sqrt{(\sin\alpha)}} \exp[\pm \iota\Delta],$$

where

$$\Delta = \frac{2\pi r_1}{\lambda}[\tan\alpha - \alpha]. \tag{6·1}$$

In combining the two branches to represent the complete solution given by the sum of the upgoing and downcoming waves, we must take into account the phase change which occurs on integration round the branch point at $r = r_1$. This phase change has already been defined as $\Omega = 2q\pi$, where it is measured in going from the upgoing to the downcoming branch. Thus, since the negative sign corresponds to the upgoing branch, we have

$$\psi_1(r) = \frac{1}{r\sqrt{(\sin\alpha)}}[\exp(-\iota\Delta)+\exp(\iota\Delta+\iota 2q\pi)],$$

which may be written
$$\psi_1(r) = \frac{2\exp(\iota q\pi)}{r\sqrt{(\sin\alpha)}}\cos(\Delta+q\pi). \tag{6·2}$$

The similarity between this form of $\psi_1(r)$ and the Debye approximation to the Hankel function in Watson's analysis, as quoted by Wwedensky, is obvious.

Considering the value of Δ in (6·1), we again replace $\tan\alpha-\alpha$ by $\frac{1}{3}\alpha^3$, so that

$$\Delta = \frac{2\pi r_1}{\lambda}\tfrac{1}{3}\alpha^3 \doteq \tfrac{1}{3}x\alpha^3. \tag{6·3}$$

The numerical treatment of the problem is simplified by introducing a number of subsidiary quantities. First of all we write

$$\frac{r_0}{r} = 1-\gamma, \tag{6·4}$$

so that γ is a function of the height above the ground, and varies from 0 to 1 as r increases from r_0 to ∞. In practice, however, we are only concerned with heights above the ground which are small compared with r_0, for which γ is very small. Putting (6·4) into the value of $\cos\alpha$ from (3·7), we get

$$\cos\alpha = (1-\gamma)(1+\xi),$$

and since γ and ξ are both small, we have

$$\alpha = [2(\gamma-\xi)]^{\frac{1}{2}}. \tag{6·5}$$

Next we make the further substitution

$$\gamma-\xi = le^{\iota\delta}, \tag{6·6}$$

i.e.
$$le^{\iota\delta} = \gamma-|\xi|e^{-\iota(\frac{1}{3}\pi-\omega)}.$$

Then it follows that
$$\tan\delta = \frac{|\xi|\sin(\frac{1}{3}\pi-\omega)}{\gamma-|\xi|\cos(\frac{1}{3}\pi-\omega)}, \tag{6·7}$$

and
$$l = |\xi|\sin(\tfrac{1}{3}\pi-\omega)\operatorname{cosec}\delta. \tag{6·8}$$

When $r = r_0$, $\gamma = 0$ and $\delta = \frac{2}{3}\pi+\omega$, and as r increases δ decreases.

Now from (6·5), (6·6), and (6·8)

$$\tfrac{1}{3}\alpha^3 = \tfrac{1}{3}(2le^{\iota\delta})^{\frac{3}{2}} = \tfrac{1}{3}[2|\xi|\sin(\tfrac{1}{3}\pi-\omega)\operatorname{cosec}\delta\, e^{\iota\delta}]^{\frac{3}{2}}.$$

Thus from (6·3), by substituting for $|\xi|$ in terms of $|\rho|$,

$$\Delta = \tfrac{1}{3}[2|\rho|\sin(\tfrac{1}{3}\pi-\omega)\operatorname{cosec}\delta\, e^{\iota\delta}]^{\frac{3}{2}}. \tag{6·9}$$

Δ thus depends solely on the parameter δ, which is a real positive angle, and is determined by writing γ from (6·7) as

$$\gamma = |\xi| \cos(\tfrac{1}{3}\pi - \omega)\,[1 + \tan(\tfrac{1}{3}\pi - \omega)\cot\delta].$$

By expressing the height h above the ground as

$$h = r - r_0 \doteqdot r_0\gamma,$$

we get
$$h = r_0|\xi| \cos(\tfrac{1}{3}\pi - \omega)\,[1 + \tan(\tfrac{1}{3}\pi - \omega)\cot\delta]. \tag{6·10}$$

From (5·13) this may also be written

$$h = h_1[1 + \tan(\tfrac{1}{3}\pi - \omega)\cot\delta], \tag{6·11}$$

from which we see that when h has increased to h_1, δ has decreased to $\tfrac{1}{2}\pi$.

For a given set of conditions δ is thus a connecting parameter between h and the height function Δ.

If Δ_0 is the value of Δ at the surface of the earth, we may put $\delta = \delta_0 = \tfrac{2}{3}\pi + \omega$ in (6.9), but we may also put $\alpha = \alpha_0$ in (6·3), giving

$$\Delta_0 = \tfrac{1}{3}x\alpha_0^3 = -\tfrac{1}{3}(2\rho)^{\frac{3}{2}} \tag{6·12}$$

by putting in the value of α_0 from (3·16).

On the ground, where $\psi_1(r) = \psi_1(r_0)$, we have from (6·2)

$$\psi_1(r_0) = \frac{2\exp(\iota q\pi)}{r_0\sqrt{(\sin\alpha_0)}}\cos(\Delta_0 + q\pi),$$

so that
$$\frac{\psi_1(r)}{\psi_1(r_0)} = \frac{r_0}{r}\sqrt{\left(\frac{\sin\alpha_0}{\sin\alpha}\right)}\frac{\cos(\Delta + q\pi)}{\cos(\Delta_0 + q\pi)}. \tag{6·13}$$

From this relation we can express the gain of the level of $\psi_1(r)$ over the level of $\psi_1(r_0)$ at the surface of the earth. The factor r_0/r is very nearly unity for all practical values of r, and may be omitted in considering numerical applications of the formula. If D is the decibel gain represented by the ratio of $\psi_1(r)/\psi_1(r_0)$ in (6·13), we can split it up into the sum of two terms, i.e.

$$D = 20\log_{10}\left[\frac{\psi_1(r)}{\psi_1(r_0)}\right] = D_1 + D_2, \tag{6·14}$$

where
$$D_1 = 20\log_{10}\left[\left|\sqrt{\frac{\sin\alpha_0}{\sin\alpha}}\right|\right], \tag{6·15}$$

and
$$D_2 = 20\log_{10}\left[\left|\frac{\cos(\Delta + q\pi)}{\cos(\Delta_0 + q\pi)}\right|\right], \tag{6·16}$$

where we are, of course, assuming that the first term of the diffraction formula is predominant.

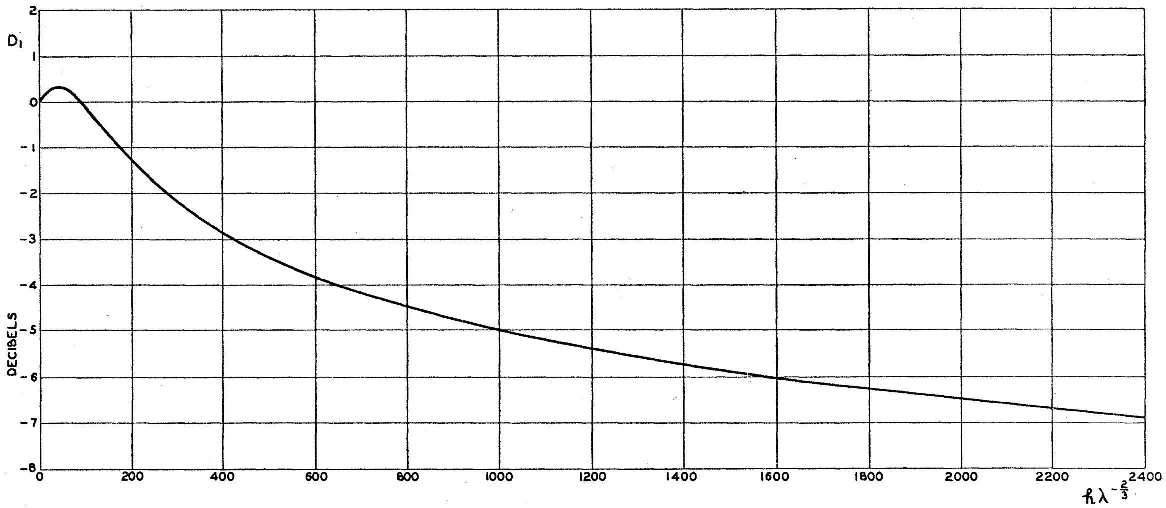
The term D_1 is small except for great heights. Since α and α_0 are small, we may write

$$\sqrt{\frac{\sin\alpha_0}{\sin\alpha}} = \left[\frac{\alpha_0}{\alpha}\right]^{\frac{1}{2}},$$

but from (6·5) and (6·6) $$\left|\frac{\alpha_0}{\alpha}\right| = \left|\frac{l_0}{l}\right|^{\frac{1}{2}} = \left[\frac{\operatorname{cosec}\delta_0}{\operatorname{cosec}\delta}\right]^{\frac{1}{2}}$$

from (6·8). Thus from (6·15) $$D_1 = 5\log_{10}\left[\frac{\sin\delta}{\sin\delta_0}\right],$$

where $\delta_0 = \frac{2}{3}\pi + \omega$, as above.



D_1 as a function of $h\lambda^{-\frac{2}{3}}$. h and λ in metres

FIG. 6

As h increases from 0 to h_1, δ decreases from $\frac{2}{3}\pi+\omega$ to $\frac{1}{2}\pi$, and $\sin\delta$ increases from $\sin(\frac{2}{3}\pi+\omega)$ to 1. As h further increases to $2h_1$, δ decreases to $\frac{1}{3}\pi-\omega$, and $\sin\delta$ decreases again to $\sin(\frac{1}{3}\pi-\omega) = \sin(\frac{2}{3}\pi+\omega) = \sin\delta_0$, whence D_1 is again zero. Over the range $h = 0$ to $2h_1$, D_1 is therefore positive, and if we take $\omega = 0$, the maximum positive value of D_1 is about 0·3 decibel. Above $h = 2h_1$, D_1 becomes negative, and reaches a value of about -5 decibels when $h = 21h_1$. Thus within the range of height with which we are concerned, D_1 is relatively small compared with the changes due to the term D_2. D_1 is only a function of δ, which in turn from (6·11) and (5·15) is seen to depend only on $h\lambda^{-\frac{2}{3}}$ for all wave-lengths for which $|\rho| = \rho_\pi$. In fig. 6, D_1 is therefore plotted as a function of $h\lambda^{-\frac{2}{3}}$, where h and λ are in metres, for use in ultra-short wave calculations.

Considering now the value of D_2 we take the value of Δ_0 in (6·12), and from (3·11) we get

$$\Delta_0 = \tfrac{1}{3}x\alpha_0^3 = -s\pi - \tfrac{1}{2}\Omega - \tfrac{1}{2}\chi,$$

and using (4·2) and (4·5),

$$\Delta_0 + q\pi = -s\pi - \tan^{-1}\eta$$

and $$\tan(\Delta_0 + q\pi) = -\eta. \tag{6·17}$$

We now write $$\Delta+q\pi = (\Delta-\Delta_0)+(\Delta_0+q\pi),$$

so that $$\frac{\cos(\Delta+q\pi)}{\cos(\Delta_0+q\pi)} = \cos(\Delta-\Delta_0)-\tan(\Delta_0+q\pi)\sin(\Delta-\Delta_0)$$
$$= \cos(\Delta-\Delta_0)+\eta\sin(\Delta-\Delta_0). \tag{6·18}$$

Initially on the ground, where $\Delta=\Delta_0$, the second term is zero and the first term is unity, but as $\Delta-\Delta_0$ increases, the second term eventually becomes predominant for short waves for which $\eta \gg 1$.

From (6·9) $\Delta-\Delta_0$ is given by

$$\Delta-\Delta_0 = \tfrac{1}{3}[2\,|\rho|\sin(\tfrac{1}{3}\pi-\omega)]^{\frac{3}{2}}[\operatorname{cosec}^{\frac{3}{2}}\delta\, e^{\iota\frac{3}{2}\delta}-\operatorname{cosec}^{\frac{3}{2}}\delta_0\, e^{\iota\frac{3}{2}\delta_0}] = A+\iota B, \tag{6·19}$$

where $$A = \tfrac{1}{3}[2\,|\rho|\sin(\tfrac{1}{3}\pi-\omega)]^{\frac{3}{2}}[\operatorname{cosec}^{\frac{3}{2}}\delta\cos\tfrac{3}{2}\delta-\operatorname{cosec}^{\frac{3}{2}}\delta_0\cos\tfrac{3}{2}\delta_0]$$

and $$B = \tfrac{1}{3}[2\,|\rho|\sin(\tfrac{1}{3}\pi-\omega)]^{\frac{3}{2}}[\operatorname{cosec}^{\frac{3}{2}}\delta\sin\tfrac{3}{2}\delta-\operatorname{cosec}^{\frac{3}{2}}\delta_0\sin\tfrac{3}{2}\delta_0].$$

We then use the expansions for $\cos(\Delta-\Delta_0)$ and $\sin(\Delta-\Delta_0)$, and remembering that η is given by (4·14), the results can be combined to determine the modulus of

$$\frac{\cos(\Delta+q\pi)}{\cos(\Delta_0+q\pi)}.$$

By writing $\eta = |\eta|\, e^{\iota\Phi}$ a rather long reduction gives

$$\left|\frac{\cos(\Delta+q\pi)}{\cos(\Delta_0+q\pi)}\right| = \{(\cos^2 A+\sinh^2 B)+|\eta|\,(\cos\Phi\sin 2A-\sin\Phi\sinh 2B)$$
$$+|\eta|^2(\sin^2 A+\sinh^2 B)\}^{\frac{1}{2}}. \tag{6·20}$$

Initially when $\Delta=\Delta_0$ and $A=B=0$, this reduces to unity as it should, and when B gets large it approximates to $e^B f(|\eta|)$, where

$$f(|\eta|) = \tfrac{1}{2}[1-2\,|\eta|\sin\Phi+|\eta|^2]^{\frac{1}{2}},$$

and is equal to $\frac{1}{2}|\eta|$ when $|\eta| \gg 1$. Under these conditions B itself approximates to

$$\tfrac{1}{3}[2\,|\rho|\sin(\tfrac{1}{3}\pi-\omega)]^{\frac{3}{2}}\operatorname{cosec}^{\frac{3}{2}}\delta\sin\tfrac{3}{2}\delta,$$

so that D_2 in (6·16) becomes

$$D_2 = 20\log_{10}[f(|\eta|)]+D_3, \tag{6·21}$$

where $$D_3 = [20\log_{10}e]\,[\tfrac{1}{3}\{2\,|\rho|\sin(\tfrac{1}{3}\pi-\omega)\}^{\frac{3}{2}}\operatorname{cosec}^{\frac{3}{2}}\delta\sin\tfrac{3}{2}\delta].$$

D_3 corresponds to $|\exp(-\iota\Delta)|$, i.e. when the upgoing wave predominates. Assuming that $\omega=0$, D_3 becomes

$$D_3 = [20\log_{10}e]\,[\tfrac{1}{3}(\sqrt{3}\,|\rho|)^{\frac{3}{2}}\operatorname{cosec}^{\frac{3}{2}}\delta\sin\tfrac{3}{2}\delta],$$

which may be written $$D_3 = |\rho|^{\frac{3}{2}} f(\delta), \tag{6·22}$$

where $$f(\delta) = 6{\cdot}60\operatorname{cosec}^{\frac{3}{2}}\delta\sin\tfrac{3}{2}\delta. \tag{6·23}$$

Now by using (5·14) and putting $\omega = 0$, (6·11) can be written

$$h = |\rho| \lambda^{\frac{2}{3}} \phi(\delta), \tag{6·24}$$

where $$\phi(\delta) = 27{\cdot}2[1+\sqrt{3}\cot\delta]. \tag{6·25}$$

$f(\delta)$ and $\phi(\delta)$ can both be plotted against δ, and hence a graph of $f(\delta)$ against $\phi(\delta)$ can be constructed. For a given height h, $\phi(\delta)$ can be determined from (6·24), and hence the corresponding value of $f(\delta)$ is read off the graph. From this value D_3 is found from (6·22).

For points well above h_1, where δ is small, we can write

$$\operatorname{cosec}^{\frac{3}{2}}\delta \sin\tfrac{3}{2}\delta = \tfrac{3}{2}\delta^{-\frac{1}{2}} \quad \text{and} \quad \cot\delta = \delta^{-1},$$

whence from (6·23) and (6·25)

$$\phi(\delta) = 27{\cdot}2 + 0{\cdot}482[f(\delta)]^2.$$

As δ gets very small the second term predominates, and

$$\phi(\delta) = 0{\cdot}482[f(\delta)]^2.$$

Substitution in this equation from (6·21) and (6·23) gives on reduction

$$D_3 = 1{\cdot}44\,|\rho|\,h^{\frac{1}{2}}\lambda^{-\frac{1}{3}}.$$

This agrees with the value found in (5·12). It is obvious that the relation is only true for points well above $h = h_1$, for in the previous argument h was height measured from $|r_1|$, and is here measured from the ground. But we see from the more accurate analysis that D_3 is a function of $h\lambda^{-\frac{2}{3}}$ for all values for which the upgoing wave predominates, since from (6·22) it depends on $f(\delta)$, and as we have seen δ depends on $h\lambda^{-\frac{2}{3}}$. Moreover for ultra-short waves, for which $|\rho| = \rho_\pi$, we can plot a fundamental graph of D_3 against $h\lambda^{-\frac{2}{3}}$, which can be used for all such wave-lengths.

Having considered the region above h_1, we proceed to study the height-gain relation close to the earth, where both branches have to be taken into account. We have to return to the full relation in (6·20), and we see that initially the $|\eta|$ term predominates over the $|\eta|^2$ term. Now it will be found that in the coefficient of $|\eta|$, the terms in $\Phi \sinh 2B$ is initially greater than the term $\cos\Phi \sin 2A$, since Φ is of the order of $\frac{5}{12}\pi$, i.e. is nearly $\frac{1}{2}\pi$. Thus the $|\eta|$ term starts by being negative, and the modulus of the whole expression actually becomes less than unity. This means that on going up from the ground, there is first of all a diminution in signal strength. This effect has been noted by Wwedensky in a particular case chosen to illustrate his analysis. The nature and magnitude of the effect can be seen from the following analysis:

Suppose that for a small height h above the ground δ has become $\frac{2}{3}\pi + \omega - \delta'$, where δ' is a small positive angle. We can now expand the expression for h in (6·10) to the first power in δ', and it will be found to reduce to

$$h = r_0\,|\rho|\,x^{-\frac{2}{3}}\delta' \operatorname{cosec}(\tfrac{1}{3}\pi - \omega). \tag{6·26}$$

By a rather long reduction $\Delta - \Delta_0$ in (6·19) becomes

$$\Delta - \Delta_0 = \tfrac{1}{2}[2\rho]^{\frac{3}{2}} \operatorname{cosec}(\tfrac{1}{3}\pi - \omega)\, e^{\iota(\frac{1}{3}\pi-\omega)}\, \delta'.$$

Putting $\cos(\Delta - \Delta_0) = 1$, and $\sin(\Delta - \Delta_0) = \Delta - \Delta_0$, since δ' is small, and replacing η by its value in (4·14), we can write (6·18) on reduction as

$$\frac{\cos(\Delta + q\pi)}{\cos(\Delta_0 + q\pi)} = 1 + \frac{x^{\frac{1}{3}}}{|\zeta|} |\rho|\, \delta' \operatorname{cosec}(\tfrac{1}{3}\pi - \omega)\, e^{\iota(\frac{1}{2}\pi - v)},$$

and substituting for $|\rho|\,\delta' \operatorname{cosec}(\frac{1}{3}\pi - \omega)$ from (6·26) and using (3·5) this becomes

$$\frac{\cos(\Delta + q\pi)}{\cos(\Delta_0 + q\pi)} = 1 + h\frac{2\pi}{\lambda}\frac{e^{\iota(\frac{1}{2}\pi - v)}}{|\zeta|}. \tag{6·27}$$

Now even if we do not assume that v is $-\frac{1}{4}\pi$, its value will be negative, so that $\frac{1}{2}\pi - v$ is $> \frac{1}{2}\pi$. Thus the modulus of the whole expression is initially less than unity for very small values of h. If we take $v = -\frac{1}{4}\pi$ and $|\zeta| = \sqrt{(2\sigma\lambda c)}$, then

$$\frac{\cos(\Delta + q\pi)}{\cos(\Delta_0 + q\pi)} = 1 + \frac{h\sqrt{2}\,\pi}{\lambda^{\frac{3}{2}}\sqrt{(\sigma c)}}\, e^{\iota\frac{3}{4}\pi}.$$

In fig. 7 this expression is represented vectorially. AB represents the unit vector along the zero direction, and BX represents the direction $\frac{3}{4}\pi$. As the length of the vector

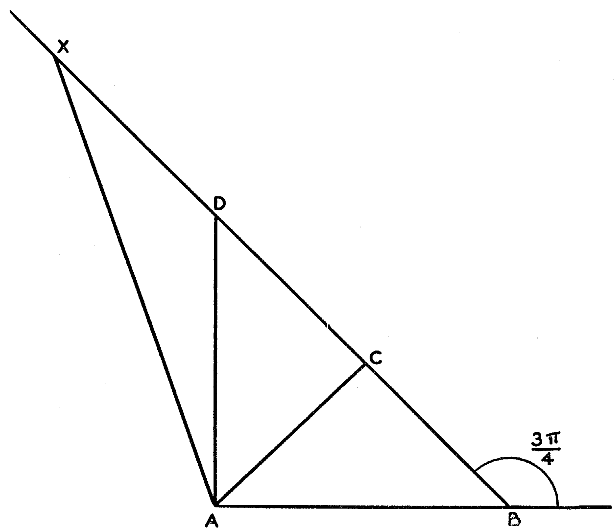


FIG. 7

along BX corresponding to the second term increases from zero with h, it is clear that the resultant vector initially decreases, and reaches a minimum of $1/\sqrt{2}$ at AC, when BC is also $1/\sqrt{2}$. If $h_{\min}$ is the corresponding value of h, then

$$h_{\min} = \frac{\sqrt{(\sigma c)}}{2\pi}\lambda^{\frac{3}{2}}. \tag{6·28}$$

The minimum resultant corresponds to a drop of 3 decibels. The resultant is again unity at AD when $h = 2h_{\min}$, and when plotted against h between 0 and $2h_{\min}$, is symmetrical about the value at $h_{\min}$. So long as our approximations involving δ' are valid, the value of the resultant AX for any given value of h is given by

$$AX = \left[1 - \frac{h}{h_{\min}} + \tfrac{1}{2}\left(\frac{h}{h_{\min}}\right)^2\right]^{\frac{1}{2}},$$

and from this expression the initial shape of the height-gain curves, when $v = -\tfrac{1}{4}\pi$, can be determined as a function of $h/h_{\min}$.

If $h_{\min}$ and λ are expressed in metres and σ in e.m.u., then (6·28) becomes

$$h_{\min} = \frac{10^6\sqrt{(3\sigma)}}{2\pi}\lambda^{\frac{3}{2}} \text{ metres.}$$

For $\lambda = 10$ m. and $\sigma = 10^{-11}$, $h_{\min} = 27{\cdot}8$ m.

It is interesting to note that (6·28) is independent of ρ, and is therefore the same for all the terms of the diffraction formula.

To calculate the complete height-gain curve between $h = 0$ and h_1 we must return again to the general expression in (6·20). This is a long but straightforward process, and the curve can be worked out as far as required, i.e. until it approximates to the curve obtained by using the upgoing wave only, where the form of D_2 is given by (6·21). At these greater heights the total gain D from (6·14) is given by

$$D = D_1 + D_3 + 20\log_{10}[f(|\eta|)], \tag{6·29}$$

and, as we have seen, for short waves D_1 and D_3 are both functions of $h\lambda^{-\frac{2}{3}}$ only. We can thus plot $D_1 + D_3$ as a function of $h\lambda^{-\frac{2}{3}}$. The form of D below the region where the upgoing wave predominates and (6·29) applies, depends on the value of σ and λ, and on the value of ϵ if we cannot assume that $2\sigma\lambda c \gg \epsilon$. But we can plot the results for practical use on a single graph in the following way:

We choose a given number of particular cases, e.g. $\lambda = 2$, 4, 6, 8 and 10 m., to cover the interesting range 2 to 10 m., and take $\epsilon = 5$ and $\sigma = 10^{-13}$ for over land, and $\epsilon = 80$ and $\sigma = 10^{-11}$ for over sea. We then evaluate D_2 from (6·16) and (6·20), and plot for each case a curve of

$$D_1 + D_2 - 20\log_{10}[f(|\eta|)] \quad \text{against} \quad h\lambda^{-\frac{2}{3}}.$$

These curves will eventually all join up into the single curve for $D_1 + D_3$ where the upgoing wave predominates. We thus get a single curve fanning out into a series of curves as $h\lambda^{-\frac{2}{3}}$ approaches zero, as is shown in figs. 8 and 9. In using these curves we have to remember to add on the value of $20\log_{10}[f(|\eta|)]$ appropriate to the wave-length and conditions we are considering, and to help in doing this the values to be added on in this way are tabulated on the graph. In computing these curves ρ was assumed equal to ρ_π in all cases, but account was taken of the fact that it is not really justifiable to assume that $2\sigma\lambda c$ is $\gg \epsilon$, e.g. for $\lambda = 2$ m. and $\sigma = 10^{-13}$, $\epsilon = 5$ and $2\sigma\lambda c = 1{\cdot}2$. The

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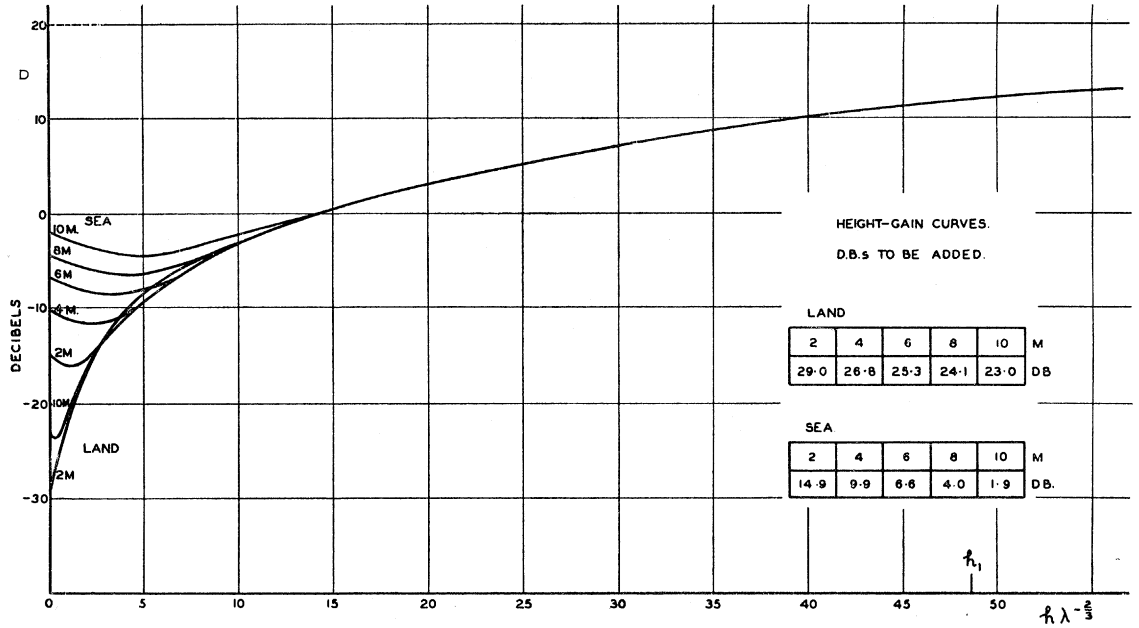
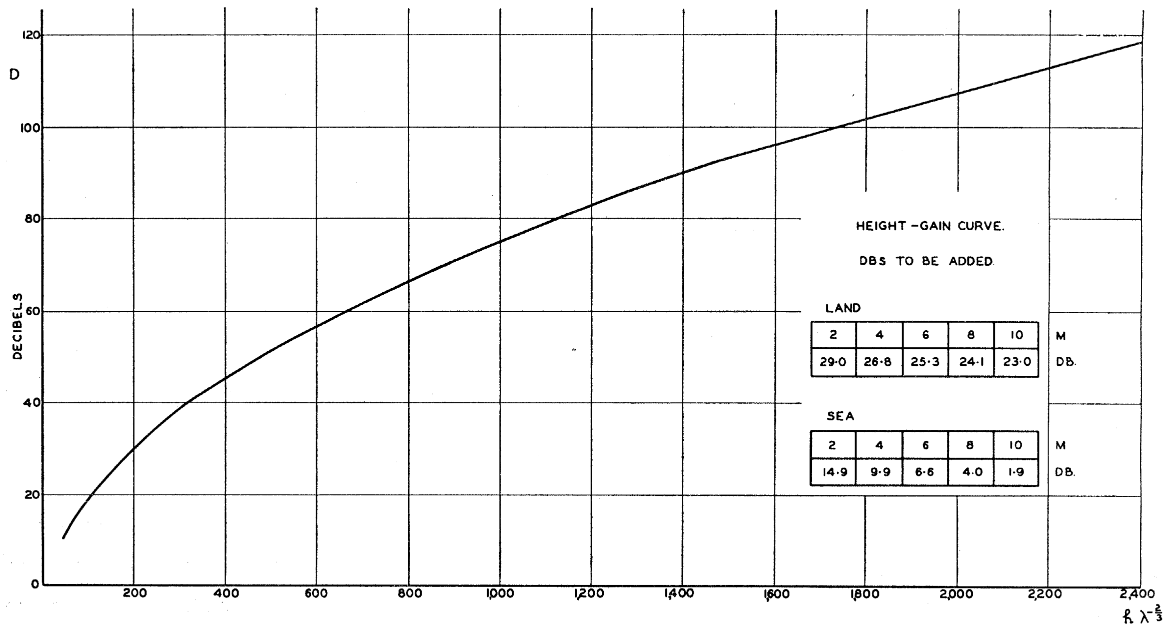


h and *λ* in metres

FIG. 8



h and *λ* in metres

FIG. 9

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

phase angle of ζ is no longer nearly $-\frac{1}{4}\pi$, but is much nearer zero. Referring again to fig. 7, the direction of BX instead of being $\frac{3}{4}\pi$ is now $\frac{1}{2}\pi - v$ from (6·27), i.e. is $\frac{1}{2}\pi + v'$, where v' is a positive angle considerably less than $\frac{1}{4}\pi$. The minimum resultant is now $\cos v'$ instead of $1/\sqrt{2}$, and the initial drop is much less, with a correspondingly smaller value of h_{min}. On the scale on which the curves are drawn in fig. 8 the effect is therefore negligible, except on the longer waves over sea, and we would not expect to be able to detect the effect experimentally.

Since $|\eta|$ increases as σ and λ decrease, the initial parts of the curves in fig. 8 get steeper as the wave-length is decreased, and the over land curves are much steeper than the over sea ones. This is equivalent to the fact that the ground attenuation due to earth losses increases rapidly with decreasing σ and λ.

By using the reciprocal theorem the analysis applies as well to the case when the receiver is on the ground and the transmitter is elevated. By combining both cases the effect of having both the transmitter and receiver elevated can be calculated. If h_T and h_R are the heights of the transmitter and receiver respectively above the ground, then the signal strength at the receiver is obtained by adding to the value worked out with both assumed on the ground, the gain reckoned from the ground up to h_R, plus the gain reckoned from the ground up to h_T. We thus get the benefit of the large initial gain close to the ground twice over, and the net gain is considerably greater than would be obtained by raising either the transmitter or the receiver alone to a height $h_T + h_R$. For a given value of $h_T + h_R$ the optimum arrangement, if practical, is to make $h_T = h_R$.

It is here that the importance of the fictitious radius $|r_1|$ comes in. With both the transmitter and receiver elevated it is necessary for the line joining them to clear the fictitious sphere of radius $|r_1|$, if the signals are to be unobstructed by the intervening earth. This means, for instance, that on 10 m. the line joining them must clear the surface of the earth by about 225 m., as will be seen from (5·15).

The absolute values of signal strength at the surface of the earth are dealt with in the next section, but assuming for the moment the results there obtained, the height-gain analysis can be used to prepare a set of graphs for various wave-lengths, giving the signal strength for 1 kW. radiated, against distance from the transmitter for various heights of the receiver above the ground. Fig. 10 shows a typical set of curves drawn for $\lambda = 10$ m., $\epsilon = 5$ and $\sigma = 10^{-13}$ e.m.u. Above the tangent plane and for distances within about 100 km., where the first term of the diffraction formula no longer predominates, the curves have been calculated by the simple theory of a direct ray and a reflected ray, using the reciprocal theorem, and taking into account the imperfect reflexion and the curvature of the earth. The diffraction curves are then joined up to these by eye, and the uncertainty in this region can only be one or two decibels.

The straight line portions of the curves show where the first term of the diffraction theory is predominant, and the fact that they are all parallel shows that the height-gain relation is independent of the distance from the transmitter. A complete set of curves for $\lambda = 2$ to 10 m. for land and sea has been published elsewhere (Eckersley 1937).

It may be noted here that there is a complete absence of an "optical range" at small heights. The earth losses are so heavy, even close to the transmitter, that the curve passes almost straight over into the diffraction curve. Even at greater heights, where the curves begin comparatively flat and then bend over into the diffraction curves, there is by no means a sudden and complete cut off.

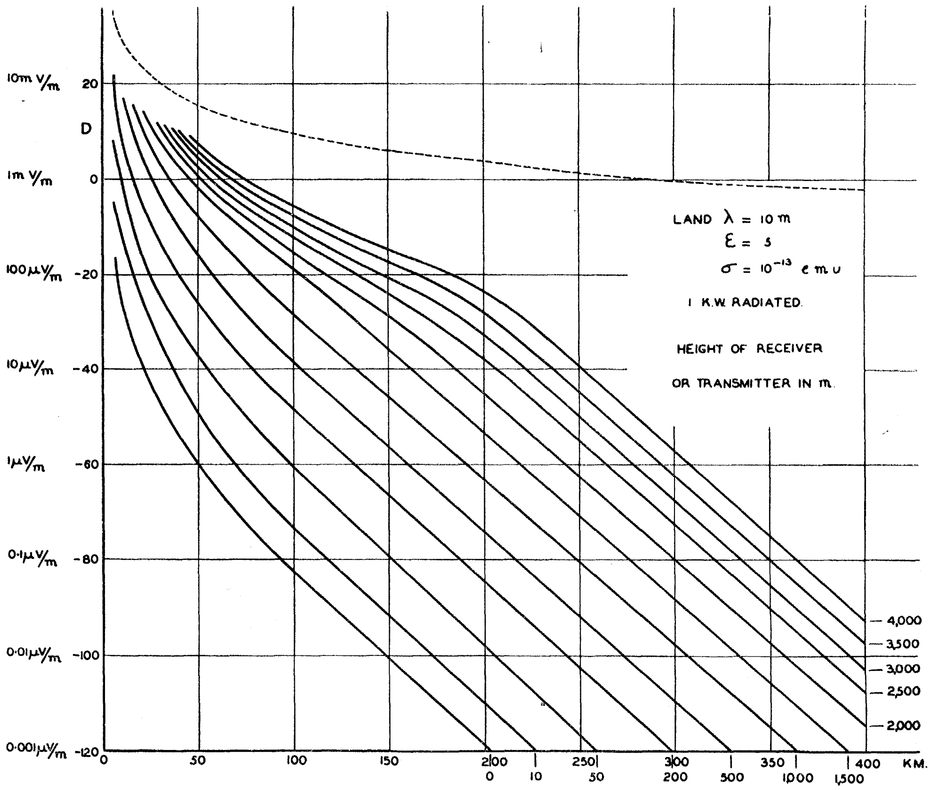


FIG. 10

7. The absolute values at the surface of the earth

The phase integral method, as developed in the above analysis, has given the general form of the potential function ψ. In order to convert the relations so that they can be used for calculating absolute values of signal strength for a given radiated power, recourse must be made to the fundamental technique on which Watson's analysis is based, namely a detailed consideration of the boundary conditions which must be fulfilled by the solution of the differential equations for ψ. This process would follow along the lines already given by Watson, the point of the phase integral method being that it provides an alternative to the very complex analysis involved in the handling of the contour integral, which, besides being simpler, exhibits the result as belonging to a type of general eigenfunction solutions.

We can, however, deduce from the phase integral theory an expression representing the variation of the amplitude factor A_s in (1·4) with wave-length and earth constants.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

This variation is expressed by the value of $\phi(s)$ in (1·2) when put into (1·1). Here, to avoid confusion, we should call this function $\phi(n)$, since Watson's s is our proper value n, and we have used s for the order of the diffraction term. As we have used ζ for another purpose needed in the present argument, we will replace Watson's ζ in (1·1) and (1·2) by ξ. Also his ka is our x. Now on the surface of the earth, where b in (1·1) is equal to a, (1·1) can be converted by using (1·2) to a form in which the ξ functions all appear in a term in the denominator; and further by approximating to Watson's ψ functions, since the proper values n are known to be large, this denominator can be written

$$\frac{d}{dn}\left[\frac{\xi'(x)}{\xi(x)}-\frac{\iota}{\zeta}\right]. \tag{7·1}$$

Moreover the proper values are determined by the equation

$$\frac{\xi'(x)}{\xi(x)}-\frac{\iota}{\zeta}=0. \tag{7·2}$$

Now returning to our proper value relation as given in (6·17), where we replace Δ_0 by its value $\frac{1}{3}x\alpha_0^3$ from (6·12),

$$\tan[q\pi+\tfrac{1}{3}x\alpha_0^3]=-\eta=-\frac{\iota}{\zeta\sin\alpha_0}$$

from (4·6), so that

$$-\sin\alpha_0\tan[q\pi+\tfrac{1}{3}x\alpha_0^3]-\frac{\iota}{\zeta}=0. \tag{7·3}$$

It thus appears that $\dfrac{\xi'(x)}{\xi(x)}$ in (7·2) is given from the phase integral by

$$\frac{\xi'(x)}{\xi(x)}=-\sin\alpha_0\tan[q\pi+\tfrac{1}{3}x\alpha_0^3]. \tag{7·4}$$

Actually the $\xi(x)$ function corresponds to our $\psi_1(r_0)$ function, and is proportional to $\dfrac{1}{\surd(\sin\alpha_0)}\cos(q\pi+\frac{1}{3}x\alpha_0^3)$. This is borne out by Wwedensky's analysis, where he derives $\xi(x)$ and $\xi'(x)$ from the Debye approximations to the Hankel functions, and shows that $\xi(x)$ is of this form, and that the proper value equation is given by (7·3). Actually he gets a difference of sign due to his use of the time factor $\exp(-\iota\omega t)$, but it can be shown that the two forms are strictly equivalent in every respect, except of course that he uses $\frac{1}{4}\pi$ where we use $q\pi$.

We need now from (7·1) the value of $\dfrac{d}{dn}\left[\dfrac{\xi'(x)}{\xi(x)}\right]$, and assuming that α_0 is small this is given from (7·4) by

$$\frac{d}{dn}\left[\frac{\xi'(x)}{\xi(x)}\right]=-\frac{d\alpha_0}{dn}\left[\tan(q\pi+\tfrac{1}{3}x\alpha_0^3)+x\alpha_0^3\sec^2(q\pi+\tfrac{1}{3}x\alpha_0^3)\right].$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

α_0 as a function of n is given from (3·9) and (3·6) by

$$\cos\alpha_0 \doteqdot 1-\tfrac{1}{2}\alpha_0^2 = (n+\tfrac{1}{2})/x,$$

so that

$$-\frac{d\alpha_0}{dn} = \frac{1}{\alpha_0 x}.$$

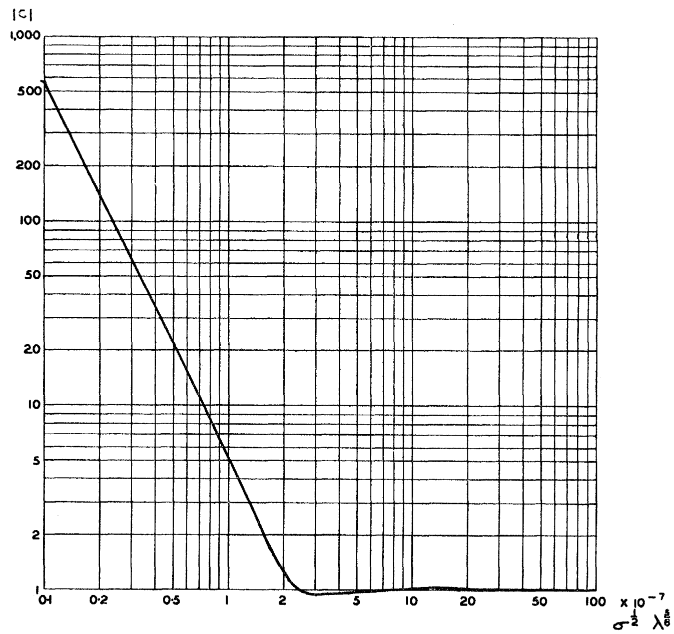
Remembering that $\tan(q\pi+\frac{1}{3}x\alpha_0^3) = -\frac{\iota}{\zeta\alpha_0}$ from (7·3), we have on reduction

$$\frac{d}{dn}\left[\frac{\xi'(x)}{\xi(x)}\right] = \alpha_0^2\left[1-\frac{1}{\zeta^2\alpha_0^2}-\frac{\iota}{\zeta x\alpha_0^4}\right]. \qquad (7\cdot5)$$

Now for perfect conductivity $\zeta=\infty$, and (7·5) reduces to α_0^2, i.e. from (3·16) to $-2\rho x^{-\frac{2}{3}}e^{-\frac{1}{3}\iota\pi}$, which is exactly the form given by Watson. It is clear, therefore, that if we write

$$C = 1-\frac{1}{\zeta^2\alpha_0^2}-\frac{\iota}{\zeta x a_0^4},$$

then C is a modification factor appearing in the denominator of the amplitude term in Watson's original formula.



σ in e.m.u.; λ in kms.

FIG. 11

Putting in the value of α_0 from (3·16) we obtain

$$C = 1+\frac{g^2}{2\rho}e^{\frac{1}{3}\iota\pi}+\frac{g}{4\rho^2}e^{\frac{1}{6}\iota\pi}, \qquad (7\cdot6)$$

where $$g = x^{\frac{1}{3}}/\zeta,$$

and if $$2\sigma\lambda c \gg \epsilon, \quad v = -\tfrac{1}{4}\pi,$$

then $$|g| = \frac{x^{\frac{1}{3}}}{|\zeta|} = \frac{(2\pi r_0)^{\frac{1}{3}}}{\sqrt{(2c)}}\sigma^{-\frac{1}{2}}\lambda^{-\frac{5}{6}}$$

and $$C = 1 - \frac{|g|^2}{2\rho}e^{-\frac{1}{6}\iota\pi} + \frac{|g|}{4\rho^2}e^{\frac{5}{12}\iota\pi}.$$

It is obvious that we can plot both the modulus and phase of C as functions of $\sigma^{\frac{1}{2}}\lambda^{\frac{5}{6}}$, since ρ has already been so represented. The value of $|C|$ plotted for $s = 0$ is shown in fig. 11.

It is interesting to note that by studying the value of $|C|$ for very small values of $|g|$ it can be shown that $|C|$ initially becomes greater than unity, but then it turns and becomes slightly less than unity before finally proceeding to the larger and larger values obtained as $\sigma^{\frac{1}{2}}\lambda^{\frac{5}{6}}$ gets smaller. The effect is similar to that noted for β near to the lower limit.

Putting the value of C into the Watson formula with its modified exponential attenuation factor, the field strength in mV./m., at a distance d km. from a vertical dipole on the surface of the earth radiating 1 kW. on a wave-length λ km. can be represented by

$$E = \frac{27\cdot5}{|C|\,|\rho|\,d^{\frac{1}{2}}\lambda^{\frac{1}{6}}}\exp\left[\frac{-0\cdot000157}{\lambda^{\frac{1}{3}}}\beta d\right]\text{mV./m.,} \qquad (7\cdot7)$$

where it is assumed that the first term of the formula is predominant.

For short waves the second term in (7·6) is predominant, and

$$|C| = \frac{|g|^2}{2|\rho|} = \frac{x^{\frac{2}{3}}}{|\zeta|^2 2|\rho|} = \frac{(2\pi r_0)^{\frac{2}{3}}\lambda^{-\frac{2}{3}}}{|\zeta|^2 2|\rho|};$$

whence from (7·7)

$$E = 0\cdot047\frac{|\zeta|^2\lambda^{\frac{1}{2}}}{d^{\frac{1}{2}}}\exp\left[\frac{-0\cdot000157}{\lambda^{\frac{1}{3}}}\beta d\right]\text{ mV./m.,}$$

where, as before, d and λ are in kilometres.

Close to the transmitter the signal strength can be calculated by Sommerfeld's theory for a flat earth, and the transition from this curve to the diffraction curve can be put in by eye. It is found that in all cases the single term diffraction value lies below the Sommerfeld curve as it should do. As we approach the transmitter, the point at which the true curve departs from the single diffraction curve may be determined approximately by examining when the second term begins to weigh, and an idea can quickly be obtained of the way in which the true curve begins to bend up towards the Sommerfeld curve, as an aid to drawing in the transition by eye. For this purpose the curve of β for the second term of the diffraction formula given in fig. 4 will be found useful, it being remembered that the corresponding $|\rho|$ values can be found from (3·19), since ω is less than 7·5° and can be neglected in this connexion.

The ground curves for various wave-lengths from 2 to 2000 m. have been drawn, and a good fit with the Sommerfeld curve is obtained in every case. In particular, as mentioned above, the height-gain curves from 2 to 10 m. (see fig. 10) are based on ground curves calculated in this way.

8. The effect of air refraction

We come lastly to the consideration of the effect of air refraction. As was stated in the introduction, this can be studied by giving a value to the function $g(r)$ in (2·5) which has so far been taken as zero.

Physical considerations indicate that to a first approximation we may consider the gradient of refractive index to be practically constant. We seek, therefore, a form of the refractive index which will obey this condition, while making the phase integral tractable. If we therefore write (2·6) in the form*

$$\mu^2 = 1 - g(r) = 1 - \eta + \epsilon \frac{r_0^2}{r^2}, \tag{8·1}$$

then

$$\frac{\partial \mu}{\partial r} = -\frac{1}{\mu} \epsilon \frac{r_0^2}{r^3}.$$

Since at the earth's surface μ is practically unity, and for small heights $r \doteq r_0$, this reduces to

$$\frac{\partial \mu}{\partial r} = -\frac{\epsilon}{r_0},$$

which is constant and negative as required.

Now if R is the radius of curvature of the ray in the atmosphere

$$\frac{1}{R} = -\frac{1}{\mu}\frac{\partial \mu}{\partial r} \doteq -\frac{\partial \mu}{\partial r},$$

so that

$$R = r_0/\epsilon. \tag{8·2}$$

From the known constitution of the lower levels of the atmosphere R is of the order of $5r_0$. Thus ϵ is of the order of 0·2. Also at the surface of the earth μ is approximately $1+2{\cdot}9\times10^{-4}$. Thus (8·1) gives

$$1-\eta+\epsilon = 1+5{\cdot}8\times10^{-4},$$

i.e.

$$\epsilon-\eta = 5{\cdot}8\times10^{-4}, \tag{8·3}$$

so that η only differs from ϵ by a very small percentage.

* The symbols η and ϵ used in this section should not be confused with the same symbols used earlier in this paper.

For the new proper value relation we have from (8·1), (2·10), and (2·11)

$$2\left(\frac{2\pi}{\lambda}\right)\int_{r_0}^{r_1}\left\{1-\eta+\epsilon\frac{r_0^2}{r^2}-\left(\frac{\lambda}{2\pi r}\right)^2 n(n+1)\right\}^{\frac{1}{2}} dr = 2\pi s+\Omega+\chi, \tag{8·4}$$

and
$$1-\eta+\epsilon\frac{r_0^2}{r_1^2}-\left(\frac{\lambda}{2\pi r_1}\right)^2 n(n+1) = 0.$$

The latter equation may be written

$$\left(\frac{\lambda}{2\pi r_1}\right)^2 [n(n+1)-\epsilon x^2] = 1-\eta, \tag{8·5}$$

and (8·4) becomes
$$2\left(\frac{2\pi}{\lambda}\right)(1-\eta)^{\frac{1}{2}}\int_{r_0}^{r_1}\left\{1-\left(\frac{r_1}{r}\right)^2\right\}^{\frac{1}{2}} dr = 2\pi s+\Omega+\chi.$$

Analogous to the former argument we write

$$r_1 = r_0(1+\xi') \quad \text{and} \quad \cos\alpha' = r_1/r,$$

and the integral becomes
$$2(1-\eta)^{\frac{1}{2}}\left(\frac{2\pi r_1}{\lambda}\right)(-\tfrac{1}{3}\alpha_0'^3) = 2\pi s+\Omega+\chi.$$

Provided we are near to the upper limit (since otherwise χ depends on ρ) this leads to

$$\xi' = \rho' x^{-\frac{2}{3}} e^{-\frac{1}{3}\iota\pi}, \tag{8·6}$$

in which
$$\rho' = \frac{\rho}{(1-\eta)^{\frac{1}{3}}}, \tag{8·7}$$

where ρ is the old value for $g(r) = 0$.

To obtain $(n+\frac{1}{2})$, we write (8·5) as

$$n(n+1) \doteqdot x^2[1-\eta+\epsilon+(1-\eta)\,2\xi'].$$

But from (8·3), $\epsilon-\eta$ is very small compared with unity, so that this may be written

$$n+\tfrac{1}{2} \doteqdot \surd\{n(n+1)\} = x[1+(1-\eta)\,\xi'].$$

The value of $\psi_2(\phi)$ in (3·12) is now

$$\psi_2(\phi) = \exp\{-\iota x[1+(1-\eta)\,\xi']\,\phi\},$$

and the attenuation factor is $\quad \exp[x(1-\eta)\,\phi[\xi']_I].$

By (8·6) this is equal to $\quad \exp[-(1-\eta)\,\rho' x^{\frac{1}{3}}\phi\sin\frac{1}{3}\pi].$

It is useful to express the attenuation in terms of the actual distance d measured along the surface of the earth, rather than of the angular distance ϕ. By writing $\phi = d/r_0$ and $x = 2\pi r_0/\lambda$, and by using (8·7), we have finally for the attenuation factor,

$$\exp\left[-\left(\frac{2\pi}{\lambda}\right)^{\frac{1}{3}}(1-\eta)^{\frac{2}{3}}\, r_0^{-\frac{2}{3}}\,\rho d \sin\tfrac{1}{3}\pi\right]. \tag{8·8}$$

Now it is obvious that this is exactly the form which would be obtained for $g(r) = 0$ on an earth of radius r_0', where

$$r_0' = \frac{r_0}{1-\eta}. \tag{8·9}$$

The effect of air refraction is thus to reduce the attenuation (as is physically obvious since the wave is bent towards the earth and tends to follow the curve of the earth), and the signal strength, obtained at a distance d, is the same as would be obtained without air refraction on an earth of radius r_0' given by (8·9).

Putting the result in another way, we may say that if the signal strength without air refraction has a certain value at a distance d, then with air refraction the signal strength has the same value at an increased distance d', where

$$d' = \frac{d}{(1-\eta)^{\frac{2}{3}}},$$

assuming that we can neglect the small change in the amplitude term so produced.

This result shows that under certain conditions, especially when there is water vapour present, the air refraction may have a large effect on the range of a short-wave transmitter. It follows similarly that under abnormal conditions, when there may be an inversion of the gradient of the refractive index due, say, to temperature, the range of a transmitter may be seriously lessened.

The relation of (8·9) is equivalent to a transformation of space, whereby the curvature of the ray produced by refraction is nullified, and the radius of the earth is increased to r_0'. Schelleng, Burrows and Ferrell (1933) were led by physical considerations to use such a transformation to determine the effect of air refraction, and the phase integral method shows that the transformation they actually used is justified by the analysis of the wave propagation.

From (8·8) we see that as η gets bigger the attenuation gets smaller. Actually it follows from (8·6) and (8·7) that when η is very nearly unity, ξ' may no longer be small, and our approximations break down. The form of (8·8) suggests, however, that when $\eta = 1$ the attenuation coefficient is zero. This we should expect physically. Since η is very nearly equal to ϵ, $\eta = 1$ corresponds from (8·2) to $R = r_0$, i.e. when $\eta = 1$ the curvature of the ray is the same as the curvature of the earth, and we should not expect any attenuation due to diffraction.

Since the effect of refraction is to increase the ground signal strength, the height-gain relation should be reduced. An argument analogous to that given previously leads to the conclusion that Δ is replaced by Δ', where

$$\Delta' = \tfrac{1}{3}[\sqrt{3}\rho \operatorname{cosec} \delta' e^{\iota\delta'}]^{\frac{3}{2}} \tag{8·10}$$

and is the same as Δ in (6·9), with $\omega = 0$, except that δ is replaced by a new parameter δ'. δ' is given in terms of a height h' by replacing (6·10), with $\omega = 0$, by

$$h' = \frac{r_0 \,|\, \xi' \,|}{2}[1+\sqrt{3}\cot\delta'],$$

i.e. from (8·6) and (8·7)

$$h' = \frac{r_0 \,|\, \xi \,|}{2(1-\eta)^{\frac{1}{3}}}[1+\sqrt{3}\cot\delta']. \tag{8·11}$$

Thus for a given decibel gain, i.e. for $\Delta = \Delta'$, we have from (8·10) that $\delta = \delta'$, and we can write (8·11) as

$$h' = \frac{h}{(1-\eta)^{\frac{1}{3}}},$$

where h' is the height at which the same gain is obtained with air refraction, as is obtained at a height h without air refraction. We thus have to go to a greater height to get the same gain, and so the height-gain relation is reduced, as was anticipated.

In practice the value of η is not sufficiently near to unity to render our assumptions invalid, so that the theory is entirely adequate to explain the effects which are to be expected on short waves. The practical application of the theory is considered in detail elsewhere (Eckersley 1937). It will suffice here to say that the theory provides an explanation for the very variable results which have been obtained on some ultra-short wave circuits, especially where the path between the transmitter and receiver has been mainly over sea, and where the receiver has been near the limit of the optical range or beyond it.

9. Conclusion

In conclusion, it may be said that the phase integral theory leads to the complete form of the solution for points on the surface of the earth, and presents the analysis in a form which exhibits clearly the physical nature of the problem. In addition, it extends the solution to points above the surface of the earth, and to the case where air refraction is present as an important factor. While the solutions are not general in the sense of applying to any point in space, however far from the earth or near to the transmitter, they provide a complete basis for calculating all the cases of practical importance. In particular the theory has been applied to the preparation of a set of ground curves for various wave-lengths from 2 to 2000 m., and of a set of height-gain curves for the range 2–10 m. It has also provided the necessary material for calculating the probable effects of air refraction.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

10. Appendix

In this paper stress has been laid upon the application of the analysis in the region where the first term of the diffraction formula predominates. But it has been pointed out that the phase integral method gives an expression for the complete solution, and as it may be useful to study in detail some cases in which it is necessary to take more than one term into account, this is given here in a form suitable as a basis for computations.

Leaving out the constant phases common to all the terms, the signal strength $|E|$ in millivolts per metre at a distance d km. from a transmitter radiating 1 kW. from a vertical dipole on the surface of the earth on a wave-length λ km. may be derived from the expression

$$E = \sum_{s=0}^{s=\infty} \frac{27\cdot 5}{C\rho d^{\frac{1}{2}}\lambda^{\frac{1}{6}}}\left(\frac{\sin\alpha_0}{\sin\alpha}\right)^{\frac{1}{2}} \frac{\cos(\Delta+q\pi)}{\cos(\Delta_0+q\pi)} \exp[-\iota\rho x^{\frac{1}{3}} e^{-\frac{1}{3}\iota\pi}\phi].$$

α and Δ are functions of the height h (in metres) of the receiver above the ground, being connected with h by the parameter δ as described in § 6. α_0 and Δ_0 are the values of α and Δ on the ground.*

q as a function of s is considered in (4·3). In (3·17) we have expressed ρ in the form $|\rho|e^{\iota\omega}$, and from (7·6) C can be expressed as $|C|e^{\iota N}$, where the phase N can also be plotted as a function of $\sigma^{\frac{1}{2}}\lambda^{\frac{5}{6}}$ when $2\sigma\lambda c \gg \epsilon$.

From § 6 $\left(\frac{\sin\alpha_0}{\sin\alpha}\right)^{\frac{1}{2}}$ can be put in the form

$$\left(\frac{\sin\delta}{\sin\delta_0}\right)^{\frac{1}{4}} e^{\iota\frac{1}{4}(\delta_0-\delta)},$$

and $\dfrac{\cos(\Delta+q\pi)}{\cos(\Delta_0+q\pi)}$ from (6·18) can be determined in the form

$$\left|\frac{\cos(\Delta+q\pi)}{\cos(\Delta_0+q\pi)}\right| e^{\iota P},$$

where the modulus has been considered in detail in (6·20).

The exponential term can be written

$$\exp[-|\rho|x^{\frac{1}{3}}\phi\sin(\tfrac{1}{3}\pi-\omega)]\exp[-\iota|\rho|x^{\frac{1}{3}}\phi\cos(\tfrac{1}{3}\pi-\omega)],$$

and replacing $|\rho|$ in terms of β in (3·19), and putting $\phi = d/r_0$ and expressing r_0 in kilometres, we get

$$\exp\left[-\frac{0\cdot 000157}{\lambda^{\frac{1}{3}}}\beta d\right]\exp\left[-\iota\frac{0\cdot 000157}{\lambda^{\frac{1}{3}}}\beta d\cot(\tfrac{1}{3}\pi-\omega)\right].$$

* When the transmitter is also elevated we must include another factor $\left(\frac{\sin\alpha_0}{\sin\alpha}\right)^{\frac{1}{2}}\frac{\cos(\Delta+q\pi)}{\cos(\Delta_0+q\pi)}$, appropriate to the height of the transmitter above the ground.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Thus, finally, we may write $$E = \sum_{s=0}^{s=\infty} |E_s| e^{\iota\Theta_s},$$

where $$|E_s| = \frac{27 \cdot 5}{|C||\rho| d^{\frac{1}{2}} \lambda^{\frac{1}{6}}} \left(\frac{\sin\delta}{\sin\delta_0}\right)^{\frac{1}{4}} \left|\frac{\cos(\Delta + q\pi)}{\cos(\Delta_0 + q\pi)}\right| \exp\left[-\frac{0 \cdot 000157}{\lambda^{\frac{1}{3}}} \beta d\right]$$

and $$\Theta_s = -N - \omega + \tfrac{1}{4}(\delta_0 - \delta) + P - \frac{0 \cdot 000157}{\lambda^{\frac{1}{3}}} \beta d \cot(\tfrac{1}{3}\pi - \omega).$$

This expression is equivalent to that obtained by Dr B. Van der Pol and Dr H. Bremmer (1937*a*, *b*), who have actually computed in some special cases up to twenty terms for distances near to the transmitter.

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